EQUIVARIANT DE RHAM COHOMOLOGY, INTEGRATION, AND LOCALIZATION: A WHIRLWIND TOUR

by

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Abstract

EQUIVARIANT INTEGRATION AND LOCALIZATION

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This thesis endeavours to introduce the formalism of the Cartan model of equivariant cohomology, with a focus towards its use in evaluating integrals via localization. It first develops the traditional theory of de Rham cohomology and integration on smooth manifolds, before introducing the theory of equivariant differential forms, and finally stating the Atiyah-Bott-Berline-Vergne localization formula and some of its corollaries.

Chapter 1: Introduction

Cohomological techniques have been unquestionably useful in the study of topological properties of manifolds. Given a cochain complex $(X^\bullet, \mathrm{d}^\bullet)$, i.e., a collection of R-modules $\{X^k\}_{k=-\infty}^{\infty}$ and a collection of module homomorphisms $\mathrm{d}^k : X^k \to X^{k+1}$ such that $\mathrm{d}^{k+1} \circ \mathrm{d}^k = 0$. Thus, we have the sequence:

$$\cdots \xrightarrow{\mathrm{d}^{k-1}} X^k \xrightarrow{\mathrm{d}^k} X^{k+1} \xrightarrow{\mathrm{d}^{k+1}} X^{k+2} \xrightarrow{\mathrm{d}^{k+2}} \cdots,$$

and we can construct the cohomology $H^\bullet(X^\bullet, \mathrm{d}^\bullet) := \ker(\mathrm{d}^\bullet)/\mathrm{im}(\mathrm{d}^{\bullet-1})$. Note that

Note that we often define $\mathrm{d} : \bigoplus_{k=-\infty}^{k} X^k \to \bigoplus_{k=-\infty}^{k} X^k$ by $\mathrm{d}x = \mathrm{d}^k x$ for $x \in X^k$.

One can construct a singular cohomology for a manifold M, where the cochains are duals of smooth maps from simplices to M. The differential becomes the dual of the boundary map, and thus we have a well-defined cohomology. It turns out though, that this is somewhat more useful than homology; while the singular homology is a R-module, we can define the cup product $\smile: H^k(M) \times H^l(M) \to H^{k+l}(M)$ under which $H^\bullet(M)$ becomes a graded ring. This extra algebraic structure makes cohomology notably easier to work with.

Thus, we have a graded ring topological invariant for every manifold. But how can we account for additional structure on the manifold? For instance, what if there is a Lie group G acting smoothly on M? The singular cohomology doesn't account for the G-action in any way.

This is where equivariant cohomology comes in. Just as singular (and de Rham) cohomology is a contravariant functor from the category of smooth manifolds to that of graded rings, G-equivariant cohomology is a contravariant functor from the category of smooth G-manifolds to that of graded rings.

But the algebraic aspect of both standard and equivariant cohomology is not the main focus of this thesis. Another remarkable boon of cohomology theory has been its connection to differential geometry, specifically its connection to integration on manifolds. This thesis was written firmly with the goal of expositing a remarkable group of related results called "localization theorems," which equate the integrals of equivariant cohomology classes to their restrictions to the fixed-point submanifolds. This can remarkably simplify the calculation of integrals, in some cases reducing them to just sums, in a way evocative of the residue theorem.

None of this thesis is original content. Everything here has been done before in multiple ways. What I hope to bring that is unique with this thesis, however, is the organization and level of detail in the derivations provided. Hopefully this will be accessible to those with some basic smooth manifold theory under his or her belt, and will be written in such a way that even the non-trivial steps are given enough time to follow readily.

Further Reading

Due to the fairly focused nature of this thesis, there are many aspects of equivariant cohomology theory that I give short shrift, or even fully ignore. Most notably, we ignore completely the Weil and Borel models for equivariant cohomology in favor of the (in this author's opinion) more sleek and elegant Cartan model, in exchange for restricting our attention to compact and connected Lie groups. For more resources on these two models, I recommend the main reference resources for this thesis, [Tu20, GS99]. For other treatments of equivariant cohomology, I refer the reader to [BGV92, Aud04]. For a physicist's perspective of the applications of this theory, I recommend [Ros21, Sza00].

Chapter 2: de Rham Cohomology

The first task that lies ahead of us is to see how we can place standard, singular cohomology, which we briefly described in the introduction, into a differential context. To do so, we will construct a cochain complex out of differential forms, constructions used to formalize integration on smooth manifolds.

In order to construct this cochain complex, we must first construct differential forms, objects of independent mathematical interest. To put it concisely, differential forms are alternating cotensor fields. But that clearly requires some unpacking. In this chapter, we will begin by working with the algebra of alternating cotensors on an arbitrary vector space V, before then defining differential forms and elucidating some of their properties. We then show how differential forms are the natural objects one integrates on smooth manifolds, and describe the differential-geometric generalization of Stokes' theorem, before showing how this factors into the construction of de Rham cohomology, which is in fact equivalent to standard singular cohomology.

2.1 Alternating Cotensors and the Exterior Algebra

First, we must build the algebraic tools on vector spaces necessary to construct differential forms on non-linear spaces (smooth manifolds).

Definition 2.1.1 (Alternating cotensors)**.** Let V be an arbitrary $n-$dimensional vector space over $\mathbb{R}$. Then $\alpha : V^k \to \mathbb{R}$ is an alternating $k-$cotensor on V if and only if:

(a) α is $k-$linear (α is a tensor)

(b) α gives 0 whenever two arguments are equal (α is alternating).

The second condition is equivalent to stating that the cotensor evaluates to 0 whenever the arguments are not all linearly independent. Thus, it becomes clear that there are no nontrivial $k-$cotensors for $k > n$.

Example 2.1.2. Let $V = \mathbb{R}^2$, and $f : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$, defined by

$$f(\mathbf{u}, \mathbf{v}) = \mathbf{u}^\intercal M \mathbf{v},$$

where

$$M = \begin{pmatrix} 0 & +\lambda \\ -\lambda & 0 \end{pmatrix}$$

for some $\lambda \in \mathbb{R}$. Then, f is an alternating $2-$cotensor on V.

Definition 2.1.3. The set of all $k-$cotensors on V is denoted $\bigwedge^k(V^*)$.

Example 2.1.4. $\bigwedge^1(V^*)$ is just the dual vector space, V^*, consisting of linear maps from $V \to \mathbb{R}$. Given a basis set $\{v_i\}_{i=1}^n$ for V, the basis of V^* are the maps $\{\omega^i\}_{i=1}^n$ defined such that $\omega^i(v_j) = \delta^i_j$.

Definition 2.1.5. The exterior algebra on V^*, denoted $\bigwedge(V^*)$, is defined as

$$\bigwedge(V^*) \equiv \bigoplus_{j=0}^{n} \bigwedge^j(V^*),$$

where $\bigwedge^0(V^*) \equiv \mathbb{R}$.

By definition, the exterior algebra has a graded structure.

The last thing we need to make it a proper algebra is a product.

Definition 2.1.6 (Exterior "wedge" product)**.** Let $\alpha \in \bigwedge^j(V^*)$ and $\beta \in \bigwedge^k(V^*)$. Then

$$(\alpha \wedge \beta) : V^{j+k} \to \mathbb{R}$$

is defined by

$$(\alpha \wedge \beta)(w_1, \cdots, w_{j+k}) = \frac{(k+j)!}{k!j!} \sum_{\sigma \in S_{k+j}} \operatorname{sgn}(\sigma) \Big(\alpha \left(w_{\sigma(1)}, w_{\sigma(2)}, \cdots, w_{\sigma(k)} \right)$$
$$\cdot \beta \left(w_{\sigma(k+1)}, w_{\sigma(k+2)}, \cdots, w_{\sigma(j+k)} \right) \Big).$$

First, we must show that this is a valid product, in that the wedge product of two alternating cotensors is itself an alternating cotensor.

Theorem 2.1.7.

$$\alpha \wedge \beta \in \bigwedge^{j+k}(V^*).$$

Proof. It is clear that $\alpha \wedge \beta$ is $(k+l)$-linear. Thus, we need only show that $\alpha \wedge \beta$ is 0 whenever two arguments are equal.

To show this, note that for any permutation that sends the two equal arguments to a and b, the permutation that sends them to b and a will cancel eachother out in the sum. As every permutation has another permutation that just switches a and b, the whole right hand side should cancel and go to 0. □

Now, we will state a few theorems necessary to be able to algebraically manipulate these objects.

Theorem 2.1.8. *If $\alpha \in \bigwedge^k(V^*)$ and $\beta \in \bigwedge^j(V^*)$ then*

$$\alpha \wedge \beta = (-1)^{kj} \beta \wedge \alpha.$$

Corollary 2.1.8.1. *If $\alpha \in \bigwedge^{2k+1}(V^*)$ for some $k \in \mathbb{N}$, then $\alpha \wedge \alpha = 0$.*

Note that this is **not** the case for even-degree alternating cotensors.

Theorem 2.1.9. *The wedge product is associative and multilinear.*

Theorem 2.1.10. *Let $\{\omega^i\}_{i=1}^n$ be the basis set for V^*. Then, the basis for $\bigwedge^k(V^*)$ is*

$$\{\omega^{i_1} \wedge \omega^{i_2} \wedge \cdots \wedge \omega^{i_k} | i_1, \cdots i_k \in \mathbb{Z} \cap [1, n],\ i_1 \neq i_2 \neq \cdots \neq i_k\}.$$

Example 2.1.11. Let $n = 4$ and $\alpha = \omega^1 \wedge \omega^3 + \omega^2 \wedge \omega^4$. Then

$$\begin{aligned}
\alpha \wedge \alpha &= (\omega^1 \wedge \omega^3 + \omega^2 \wedge \omega^4) \wedge (\omega^1 \wedge \omega^3 + \omega^2 \wedge \omega^4) \\
&= (\omega^1 \wedge \omega^3) \wedge (\omega^1 \wedge \omega^3) + (\omega^1 \wedge \omega^3) \wedge (\omega^2 \wedge \omega^4) \\
&\quad + (\omega^2 \wedge \omega^4) \wedge (\omega^1 \wedge \omega^3) + (\omega^2 \wedge \omega^4) \wedge (\omega^2 \wedge \omega^4) \\
&= \omega^1 \wedge (\omega^3 \wedge \omega^1) \wedge \omega^3 + \omega^1 \wedge (\omega^3 \wedge \omega^2) \wedge \omega^4 \\
&\quad + (\omega^2 \wedge \omega^4 \wedge \omega^1 \wedge \omega^3) + \omega^2 \wedge (\omega^4 \wedge \omega^2) \wedge \omega^4 \\
&= -\omega^1 \wedge \omega^1 \wedge \omega^3 \wedge \omega^3 - \omega^1 \wedge \omega^2 \wedge \omega^3 \wedge \omega^4 \\
&\quad - \omega^1 \wedge \omega^2 \wedge \omega^3 \wedge \omega^4 - \omega^2 \wedge \omega^2 \wedge \omega^4 \wedge \omega^4 \\
&= -2\omega^1 \wedge \omega^2 \wedge \omega^3 \wedge \omega^4 \neq 0.
\end{aligned}$$

2.2 Differential Forms

Now, we will take the techniques developed in the previous section, and extend them for use on smooth manifolds, simply put, manifolds that in some differentiable way locally "look like" Euclidean space. We will not define them beyond that here, but if one wishes to see an in-depth treatment of them, we direct the reader to [Lee12].

Definition 2.2.1 (Tangent and cotangent bundle)**.** Let $TM \equiv \sqcup_{p\in M} T_pM$, and $T^*M \equiv \sqcup_{p\in M} T_p^*M$. We call TM the tangent bundle, and T^*M the cotangent bundle.

These two bundles are specific cases of what is called a vector bundle, which is a space E along with a projector $\pi : E \to M$ such that E is locally difeomorphic to $M \times V$ for some vector space $V \cong \pi^{-1}(\{p\})$, called its fiber. We will not go into much more explanation of vector bundles, but note the following two statements about them in more generality.

Theorem 2.2.2 (Exterior power of a vector bundle)**.** *Given a smooth vector bundle $\pi : E \to M$ with fiber V, there exists a smooth vector bundle $\bigwedge^k(E)$ with fiber $\bigwedge^k(V)$.*

Proof. See [Tu17, 20.6-7] for proof. □

Definition 2.2.3 (Smooth sections of a vector bundle)**.** A smooth section of a vector bundle E is a smooth map $\sigma : M \to E$ such that $\pi \circ \sigma = \mathrm{id}_M$, i.e., $\pi(\sigma(p)) = p$ for all $p \in M$. The space of smooth sections of E is denoted $\Gamma(M, E)$.

Note that unlike more general fiber bundles, there will always exist at least one global section on a vector bundle, the zero section $\sigma_0 : p \to \mathbf{0}_p$, but not all vector bundles have a nowhere-zero smooth section.

We call smooth sections on TM vector fields on M.

Definition 2.2.4 (Differential forms)**.** A **differential k-form** ϖ on a smooth manifold M is an element of $\Omega^k(M) \equiv \Gamma(M, \bigwedge^k(T^*M))$.

Note that as $\bigwedge^0(T^*M) = \mathbb{R}$, $\Omega^0(M) = C^\infty(M)$, and as differential forms inherit a $C^\infty(M)$-linear wedge product defined by wedging at each point, $\Omega(M)$ is both a graded ring and a $C^\infty(M)$-module. In other words, differential forms are the alternating subspace of the dual space (as $C^\infty(M)$ modules) to the tensor algebra of vector fields.

Now we will begin to describe the space of differential forms in more concrete details.

Theorem 2.2.5 (The exterior derivative)**.** *There exists a unique linear map $\mathrm{d} : \Omega^k(M) \to \Omega^{k+1}(M)$ satisfying the following prpperties;*

(a) For a function (a 0-form) $f : M \to \mathbb{R}$, $\mathrm{d}f : T^*M \to C^\infty(M)$ is defined by $\mathrm{d}f(X) = X(f)$.

(b) $\mathrm{d}(\mathrm{d}\alpha) = 0$ for any form α.

(c) If α is a k-form, then $\mathrm{d}(\alpha \wedge \beta) = \mathrm{d}\alpha \wedge \beta + (-1)^k \alpha \wedge \mathrm{d}\beta$.

Proof. cf. [Lee12, Thm. 14.24] □

Theorem 2.2.6. *Given a neighborhood $U \subseteq M$ with smooth coordinates $\{x^i : M \to \mathbb{R}\}_{i=1}^n$, the $1-$forms $\{\mathrm{d}x^i\}_{i=1}^n$ form a smooth **co-frame** for T^*M on U. In other words, given any $p \in U$, $\{\mathrm{d}x_p^i\}_{i=1}^n$ forms a basis of T_p^*M.*

Proof. cf. [Lee12, Ex. 11.13] and [Lee12, p. 281]. □

Theorem 2.2.7. *Let M be a smooth n-manifold, and let $(U, \varphi = (x^1, x^2, \cdots, x^n))$ be a coordinate chart. For any $f \in C^\infty(M)$, in U,*

$$\mathrm{d}f = \sum_{i=1}^n \frac{\partial f}{\partial x^i} \mathrm{d}x^i.$$

Proof. To see this, note first that on $\mathrm{d}f(X) = X(f)$. Next, recall that U, $\left\{\frac{\partial}{\partial x^i}\right\}_{i=1}^n$ form a frame for TM [Lee12, Ex. 8.10.a]. As $\{\mathrm{d}x^i\}_{i=1}^n$ forms a coframe on U, we have that $\mathrm{df} = \sum_{i=1}^n g_i \mathrm{d}x^i$ for some $\{g_i\} \in C^\infty(U)$. Thus, as $\mathrm{d}x^i\left(\frac{\partial}{\partial x^j}\right) = \delta_j^i$ as a function (i.e., the constant function $p \to \delta_j^i$), We have that

$$\begin{aligned}
\mathrm{d}f\left(\frac{\partial}{\partial x^j}\right) &= \sum_{i=1}^n g_i \mathrm{d}x^i \left(\frac{\partial}{\partial x^j}\right) \\
&= \sum_{i=1}^n g_i \delta_j^i = g_i.
\end{aligned}$$

However, we also know that $\mathrm{d}f\left(\frac{\partial}{\partial x^j}\right) = \frac{\partial}{\partial x^j}(f) = \frac{\partial f}{\partial x^j}$. Thus, we have proven our assertion. □

2.3 Integration and Stokes' Theorem

Now we will begin to develop differential forms' raison-d'être : coordinate-free integration on manifolds. But first, some preliminaries.

Definition 2.3.1 (Pushforwards and pullbacks)**.** Let $F : M \to N$ be a smooth map between two smooth manifolds. Then $F_* : TM \to TN$ is defined in the following way. Given $v \in TM$, for any point $p \in N$ and function $g \in C^\infty(N)$, $F_* v$ satisfies

$$(F_* v)_{F(p)}(g) = v_p(g \circ F).$$

Next, we define $F^* : T^*N \to T^*M$ to be the linear map that, given any $p \in M$, $\omega \in T^*N$, and $v \in TM$

$$(F^*\omega)_p(v_p) = \omega_{F(p)}((F_* v)_{F(p)}).$$

It is straightforward to show that pushforwards are covariant and pullbacks are contravariant, i.e., $(F \circ G)_* = F_* \circ G_*$ and $(F \circ G)^* = G^* \circ F^*$. We can extend pullbacks to $F^* : \Lambda^k(T^*N) \to \Lambda^k(T^*M)$ by asserting that $F^*(\alpha \wedge \beta) = (F^*\alpha) \wedge (F^*\beta)$, and for when $f \in \Omega^0(N)$, $F^* f = f \circ F \in \Omega^0(M)$

On two unrelated (but later relevant) notes, we present a definition and a theorem.

Definition 2.3.2 (Velocity vectors of curves)**.** Given a smooth curve $\gamma : \mathbb{R} \to M$, we let

$$\gamma'(\tau) : \mathbb{R} \ni \tau \to \gamma_* \left(\frac{\mathrm{d}}{\mathrm{d}t}\Big|_\tau \right) \in T_{\gamma(\tau)}M$$

be the velocity vector to γ at $t = \tau$.

Theorem 2.3.3. *Given $F : M \to N$ and $\omega \in \Omega^k(N)$,*

$$F^*(\mathrm{d}\omega) = \mathrm{d}(F^*\omega)$$

Proof. We will show this in the following way: First, we will show this is the case for 0-forms, and 1-forms. Then, we will show how, as every k-form is a superposition of the wedge product of k 1-forms, this follows for k-forms from the property holding on 1-forms.

First, to show that it holds for any 0-form f, we note that $\mathrm{d}f(X) = X(f)$, so

$$(F^*(\mathrm{d}f))(Y) = (\mathrm{d}f)(F_*Y)$$

$$= (F_*Y)(f) = Y(f \circ F)$$

$$\mathrm{d}(F^*f)(Y) = Y(F^*f)$$

$$= Y(f \circ F).$$

Now, to show this works for a one form, let's consider an arbitrary one-form $\alpha = f\mathrm{d}g$, where $f, g \in C^\infty(N)$. It can be shown that every 1-form is a superposition of such forms, so it suffices to prove the theorem for this case to prove it for all one forms.

$$\mathrm{d}(F^*\alpha) = \mathrm{d}((F^*f)(F^*\mathrm{d}g))$$

$$= \mathrm{d}(F^*f) \wedge (F^*\mathrm{d}g) + (F^*f)\mathrm{d}(F^*\mathrm{d}g)$$

$$= (F^*\mathrm{d}f) \wedge (F^*\mathrm{d}g) + (F^*f)\mathrm{d}(\mathrm{d}(F^*g))$$

$$= F^*(\mathrm{d}f \wedge \mathrm{d}g)$$

$$= F^*(\mathrm{d}\alpha).$$

Now, we arrive at our inductive step. Suppose we have that for all k-forms ϖ that $\mathrm{d}(F^*\varpi) = F^*(\mathrm{d}\varpi)$. Then, as every $k+1$-form is the span of the wedge product of arbitrary 1-forms α

and k-forms ϖ, we need only show that

$$\begin{aligned}
\mathrm{d}(F^*(\alpha \wedge \varpi)) &= \mathrm{d}((F^*\alpha) \wedge (F^*\varpi))) \\
&= (\mathrm{d}(F^*\alpha)) \wedge (F^*\varpi) - (F^*\alpha) \wedge (\mathrm{d}(F^*\varpi)) \\
&= (F^*\mathrm{d}\alpha) \wedge (F^*\varpi) - (F^*\alpha) \wedge (F^*\mathrm{d}\varpi) \\
&= F^*(\mathrm{d}\alpha \wedge \varpi) - F^*(\alpha \wedge \mathrm{d}\varpi) \\
&= F^*(\mathrm{d}\alpha \wedge \varpi - \alpha \wedge \mathrm{d}\varpi) \\
&= F^*(\mathrm{d}(\alpha \wedge \varpi)).
\end{aligned}$$

Thus, with $k = 1$ as our base case, we use induction to show that all k forms with $k \geqslant 1$ satisfy the property, and as we've proven it for $k = 0$, we have all differential forms satisfy it. □

Definition 2.3.4 (Orientation forms and orientable manifolds)**.** Given an n-dimensional manifold M, an orientation form is a n-form ω that is at no point equivalent to the 0 map. If such a form exists, then M is said to be an orientable manifold. We say that (M, ω) is an oriented manifold.

A global, ordered coframe $\{\omega^i\}_{i=1}^n$ induces a global orientation form $\omega^1 \wedge \cdots \wedge \omega^n$, and a local ordered coframe (say on $U \subseteq M$) a local orientation form. Thus, by Thm. (2.2.6), every local coordinate system induces a local orientation form. If these forms can be stitched together in a smooth way, then M is orientable. Thus, if that can be done, it often becomes the orientation form of the oriented manifold.

Definition 2.3.5 (Positively and negatively oriented forms)**.** Given an oriented manifold (M, ω), any orientation form that can be written as $f\omega$ for some smooth $f : M \to \mathbb{R}_{>0\ (<0)}$ is **positively (negatively)-oriented**.

Now, we turn our attention to submanifolds. A manifold being orientable does not imply its submanifolds are as well - just consider the fact that the Möbius band, the most infamous non-orientable surface, is a submanifold of $\mathbb{R}^3$, which is evidently orientable. However, if the submanifold is orientable, we can induce an orientation after making some extra choices.

Theorem 2.3.6 (Induced orientation of submanifolds)**.** *Given an oriented n-manifold (M, ω) and an orientable k-submanifold $S \subseteq M$, if we can select a collection of vector fields $N_1, \cdots, N_{n-k}$ such that for all j,*

$$N_j \notin \iota_*(TS),$$

then

$$\iota^*((N_1, \ldots, N_{n-k}) \lrcorner \omega)$$

is an orientation form on S, where $\lrcorner$ is the interior multiplication $\Omega^n(M) \times (\Gamma(TM))^{n-k} \to \Omega^k(M)$ defined by $(V_1, \ldots, V_{n-k}) \lrcorner \omega = \omega(V_1, \ldots, V_{n-k}, \cdot, \cdot, \ldots)$. Interior multiplication by a vector field V may also be denoted ι_V when confusion with an inclusion map is unlikely.

This is a good thing to know, but it isn't all that relevant: for our purposes most orientable submanifolds will just be given their own orientation form. However, there is one place this will come in: the statement of Stokes' theorem, which specifically requires us to have an induced orientation for the boundary of a manifold (with boundary). In order to make sense of this though, we first need to introduce the notion of an outward-pointing vector field.

Definition 2.3.7 (Outward-pointing vectors and vector fields)**.** Given a smooth n-manifold with boundary M, and a point $p \in \partial M$ a vector $v \in T_pM$ is outward-pointing, if for some (and thus all) boundary coordinate chart (U, φ), the $\frac{\partial}{\partial x^n}$ component of v is negative. A vector field X on M points outward if for all $p \in \partial M$, X_p points outward.

Now that we have defined such a vector field, we have everything we need built up in order to induce an orientation on the boundary, as it is an $(n-1)$-manifold and we have a vector field.

Theorem 2.3.8 (The Stokes orientation)**.** *If M is a smooth, oriented n-manifold with boundary, then ∂M is an orientable manifold. Furthermore, given any vector fields X_1, X_2 pointing out of ∂M, then $X_1 \lrcorner\, \omega$ and $X_2 \lrcorner\, \omega$ are positively oriented with respect to eachother. This equivalence class of orientations is called the* ***Stokes' Orientation****.*

Proof. cf. [Lee12, Prop. 15.24] □

Example 2.3.9. Let D be the closed unit disk in $\mathbb{R}^2$ defined by $\{(x, y) \in \mathbb{R}^2 \ : \ x^2+y^2 \leqslant 1\}$. Then, notice that $\partial D = \{(x, y) \in \mathbb{R}^2 \ : \ x^2 + y^2 = 1\}$. Let $\iota_{\partial D} : \partial D \hookrightarrow D$ be the inclusion of ∂D into D. Notice an orientation on $\mathbb{R}^2$ induces one on D by restriction. Let $\omega := \mathrm{d}x \wedge \mathrm{d}y$ be our orientation on $\mathbb{R}^2$. Then, as $N := x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}$ is an outward-pointing vector field,

$$\begin{aligned}
\iota^*(N \lrcorner\, \mathrm{d}x \wedge \mathrm{d}y) &= \iota^* \left(\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} \right) \lrcorner\, \mathrm{d}x \wedge \mathrm{d}y \right) \\
&= \iota^* \left(\left(x\frac{\partial}{\partial x} \right) \lrcorner\, \mathrm{d}x \wedge \mathrm{d}y + \left(y\frac{\partial}{\partial y} \right) \lrcorner\, \mathrm{d}x \wedge \mathrm{d}y \right) \\
&= \iota^* \left(x\mathrm{d}y - y\mathrm{d}x \right),
\end{aligned}$$

where we used that $X \lrcorner\, (\alpha \wedge \beta) = (X \lrcorner\, \alpha) \wedge \beta + (-1)^{\deg \alpha} \alpha \wedge (X \lrcorner\, \beta)$. This is our orientation form on ∂D.

Definition 2.3.10 (Integrating differential forms on $\mathbb{R}^n$, see [Tu11, Def. 23.8] and [Lee12, p. 402])**.** Let $\{x^i\}_{i=1}^n$ be a coordinate system for $\mathbb{R}^n$, inducing the orientation form $\mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^n$, and ω be an n-form on open subset $U \subset \mathbb{R}^n$. This form is expressible as $f(\mathbf{x})\mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^n$ for some smooth function $f : \mathbb{R}^n \to \mathbb{R}$. The integral of ω over subset $A \subset U$ is defined as

$$\int_A \omega = \int_A f(\mathbf{x})\mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^n \equiv \int_A f(\mathbf{x})\mathrm{d}x^1 \cdots \mathrm{d}x^n$$

if the rightmost (Riemann) integral exists.

In other words, simply "erase the wedges," as Lee puts it.

Definition 2.3.11 (Integration of differential forms on one chart, cf. [Lee12, p. 404], [Tu11, eq. 23.7])**.** Suppose $U \subseteq M$ is an open neighborhood of an oriented n-manifold M, and U contains the compact support of an n-form ω. Further suppose that $\varphi : U \to \varphi(U) \subseteq \mathbb{R}^n$ is the coordinate diffeomorphism (i.e., each of its components is a coordinate function x^i), which induces an orientation on $\varphi(U)$. Then let the integral of ω over M be defined as

$$\int_M \omega \equiv \int_{\varphi(U)} (\varphi^{-1})^* \omega.$$

Now, one of the things about smooth manifolds is that they are only *locally* diffeomorphic to $\mathbb{R}^n$, so many n-forms will have their support lie in more than one so-called "coordinate chart." By definition, a smooth manifold has an atlas, with finite open cover $\{U_i\}_i$ and coordinate diffeomorphisms $\{\varphi_i\}_i$ such that the so-called transition maps $\varphi_i \circ \varphi_j^{-1}$ are smooth for all i and j such that $U_i \cap U_j \neq 0$. There may be differential forms with support on multiple of the U_i. To extend Def. (2.3.11), we introduce the concept of a partition of unity, a collection of smooth functions $\{\varepsilon_i\}_i$ such that $\sum_i \varepsilon_i = 1$, but each ε_i is only nonzero on U_i. It is known that for a smooth manifold such a partition must exist [Lee12, Thm. 2.23]. Thus, for any n-form ω with compact support on M, we can decompose it into $\sum_i \varepsilon_i \omega$, and, each term in the sum having compact support on only one U_i, then say that

$$\int_M \omega \equiv \sum_i \int_{\varphi_i(U_i)} (\varphi_i^{-1})^* (\varepsilon_i \omega).$$

It is known that, while this definition is built upon choices of open covers, coordinates, and partitions of unity, it does not depend on any of those choices [Lee12, Prop. 16.4-5].

It is also worth noting that if S is a smooth k-submanifold of M with inclusion $\iota_S : S \hookrightarrow$

M, and $\omega \in \Omega^k(M)$, then we often slightly abuse notation, saying that

$$\int_S \omega \equiv \int_S \iota_S^* \omega.$$

Another useful property about integrating differential forms is that they are invariant under pullbacks, which in turn gives us that differential forms are in fact coordinate-independent.

Theorem 2.3.12. *Let $f : U \subseteq M \to V \subseteq N$ be an orientation-preserving diffeomorphism between open subsets of orientable manifolds M and N, and $\omega \in \Omega(N)$. Then,*

$$\int_{M=f^{-1}(N)} f^* \omega = \int_N \omega.$$

Proof. Assume without loss of generality that U and V are coordinate charts of M and N respectively (if they are not, we can intersect them with the charts, and construct a partition of unity, etc. and the following argument will still work).

Then, if $\varphi : U \to \varphi(U) \subseteq \mathbb{R}^n$ and $\psi : V \to \psi(V) \subseteq \mathbb{R}^n$ are coordinate charts, we have that our assertion is equivalent to:

$$\int_{\varphi(U)} (\varphi^{-1})^* f^* \omega = \int_{\psi(V)} (\psi^{-1})^* \omega$$

$$\int_{\varphi(U)} (f \circ \varphi^{-1})^* \omega = \int_{\psi(V)} (\psi^{-1})^* \omega.$$

Now, as we are working in coordinate charts, and as n-forms are the only forms with non-vanishing integrals, we can write $(\psi^{-1})^* \omega = g(\mathbf{x}) \mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^n$ and $(f \circ \varphi^{-1})^* \omega = h(\mathbf{x}) \mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^n$, where they are related by

$$(\psi \circ f \circ \varphi^{-1})^* g \mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^n = h \mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^n$$

$$(g \circ \kappa) \mathrm{d}(x^1 \circ \kappa) \wedge \cdots \wedge \mathrm{d}(x^n \circ \kappa) = h \mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^n.$$

where we've written that $\kappa = \psi \circ f \circ \varphi^{-1}$ for brevity. Now, expand $\mathrm{d}\kappa^i$ out in coordinates to get that

$$(g \circ \kappa) \left(\sum_i \frac{\partial \kappa^1}{\partial x^i} \mathrm{d}x^i \right) \wedge \cdots \wedge \left(\sum_i \frac{\partial \kappa^n}{\partial x^i} \mathrm{d}x^i \right) = (g \circ \kappa) \det J \mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^n,$$

where $J_i^j = \partial_j \kappa^i$ is the Jacobian of κ. Thus, we have that our assertion is equivalent to

$$\int_{\phi(U)} g \circ \kappa \det J \mathrm{d}x^1 \cdots \mathrm{d}x^n = \int_{\psi(V)} g \mathrm{d}x^1 \cdots \mathrm{d}x^n,$$

which is true by the standard change-of-variables formula in multivariable calculus. □

Example 2.3.13 (Line integrals)**.** Let $\mathbb{R}^2$ be our ambient manifold, let $\omega = f(x,y)\mathrm{d}x + g(x,y)\mathrm{d}y$, and let $\gamma : [a,b] \to \mathbb{R}^2$ be a C^∞ curve. Then we have that

$$\begin{aligned} \int_{\gamma([a,b])} \omega &= \int_{[a,b]} \gamma^* \omega \\ &= \int_{[a,b]} \left(f(\gamma^x(t), \gamma^y(t)) \mathrm{d}\gamma^x + g(\gamma_x(t), \gamma_y(t)) \mathrm{d}\gamma^y \right). \end{aligned}$$

We are playing fast and loose with the fact that charts are supposed to be over open sets here with the knowledge that the boundary points will in the end comprise part of a set of measure zero, and as such will not matter.

Now, we evaluate $\mathrm{d}\gamma^x$ and $\mathrm{d}\gamma^y$ with Thm. (2.2.7) and get that $\mathrm{d}\gamma^x = \frac{\mathrm{d}\gamma^x}{\mathrm{d}t}\mathrm{d}t$ and $\mathrm{d}\gamma^y = \frac{\mathrm{d}\gamma^y}{\mathrm{d}t}\mathrm{d}t$, so we get the integral equal to

$$\int_{[a,b]} (f(\gamma^x(t), \gamma^y(t))(\gamma^x)'(t) + g(\gamma^x(t), \gamma^y(t))(\gamma^y)'(t))\mathrm{d}t,$$

which has taken the standard form of a line integral seen in calculus III for a vector field $f(x,y)\hat{\mathbf{x}} + g(x,y)\hat{\mathbf{y}}$. Note that we are using an orientation given by the curve in the direction of increasing t in this calculation. In practice, when we give an orientable submanifold on which we wish to integrate a form, it will come with its own orientation.

Theorem 2.3.14 (Stokes' theorem)**.** *Given an orientable smooth n-manifold M with boundary ∂M, and a differential $(n-1)$-form ω with compact support on M,*

$$\int_M \mathrm{d}\omega = \int_{\partial M} \omega.$$

Example 2.3.15 (The divergence theorem)**.** Let $V \subseteq \mathbb{R}^3$ be a 3-submanifold with closed boundary $S \equiv \partial V$. Let $\mathbf{F}(x,y,z) = F_x(x,y,z)\hat{\mathbf{x}} + F_y(x,y,z)\hat{\mathbf{y}} + F_z(x,y,z)\hat{\mathbf{z}}$ be a (Calculus III-style) vector field. Let $\{U_i, \varphi_i = (s_i,t_i)\}$ be a coordinate atlas for S onto $\varphi_i(U_i) \subseteq \mathbb{R}^2$, and ε_i a compatible partition of unity over S, $\tilde{\varepsilon}_i = \varepsilon_i \circ S_i$, $S_i \equiv \varphi_i^{-1}$, and $\varepsilon_i' \equiv \varepsilon_i \circ \iota^{-1}$, restricted to where the map is well defined. Finally, let

$$(f,g)_{(x,y)} \equiv f_x g_y - f_y g_x$$

and let $\iota : S \to \mathbb{R}^3$. be the inclusion. Let $\mathbf{r}_i = \iota \circ S_i = (r_i^x, r_i^y, r_i^z)$. Then,

$$\begin{aligned}\oiint_S \mathrm{d}\mathbf{S}\cdot\mathbf{F} &= \sum_i \iint\limits_{\varphi_i(U_i)} \mathrm{d}s_i \mathrm{d}t_i \tilde{\varepsilon}_i(((r_i)_y,(r_i)_z)_{(s_i,t_i)}(F_x\circ\mathbf{r}_i) + ((r_i)_z,(r_i)_x)_{(s_i,t_i)}(F_y\circ\mathbf{r}_i) \\ &\qquad + ((r_i)_x,(r_i)_y)_{(s_i,t_i)}(F_z\circ\mathbf{r}_i)) \\ &= \sum_i \iint\limits_{\varphi_i(U_i)} \tilde{\varepsilon}_i(((r_i)_y,(r_i)_z)_{(s_i,t_i)}(F_x\circ\mathbf{r}_i) + ((r_i)_z,(r_i)_x)_{(s_i,t_i)}(F_y\circ\mathbf{r}_i) \\ &\qquad + ((r_i)_x,(r_i)_y)_{(s_i,t_i)}(F_z\circ\mathbf{r}_i))\mathrm{d}s_i \wedge \mathrm{d}t_i\end{aligned}$$

We have that $((r_i)_x, (r_i)_y)_{(s_i,t_i)}\mathrm{d}s_i \wedge \mathrm{d}t_i = (\mathbf{r}_i)^*(\mathrm{d}x \wedge \mathrm{d}y)$, which we can see by the fact that

$$\begin{aligned}\mathbf{r}_i^*(\mathrm{d}x \wedge \mathrm{d}y) &= (\mathbf{r}_i^*\mathrm{d}x) \wedge (\mathbf{r}_i^*\mathrm{d}y)\\ &= \mathrm{d}(\mathbf{r}_i^*x) \wedge \mathrm{d}(\mathbf{r}_i^*y)\\ &= \mathrm{d}r_i^x \wedge \mathrm{d}r_i^y\\ &= \left(\frac{\partial r_i^x}{\partial s_i}\mathrm{d}s_i + \frac{\partial r_i^x}{\partial t_i}\mathrm{d}t_i\right) \wedge \left(\frac{\partial r_i^y}{\partial s_i}\mathrm{d}s_i + \frac{\partial r_i^y}{\partial t_i}\mathrm{d}t_i\right)\\ &= \left(\frac{\partial r_i^x}{\partial s_i}\frac{\partial r_i^y}{\partial t_i} - \frac{\partial r_i^y}{\partial s_i}\frac{\partial r_i^x}{\partial t_i}\right)\mathrm{d}s_i \wedge \mathrm{d}t_i.\end{aligned}$$

Similar analysis holds for the pullbacks of all the other coframe 2-forms.

Thus,

$$\begin{aligned}\oiint_S \mathrm{d}\mathbf{S}\cdot\mathbf{F} &= \sum_i \iint_{\varphi_i(U_i)} \tilde{\varepsilon}_i((F_x \circ \mathbf{r}_i)\mathbf{r}_i^*(\mathrm{d}y \wedge \mathrm{d}z) + (F_y \circ \mathbf{r}_i)\mathbf{r}_i^*(\mathrm{d}z \wedge \mathrm{d}x) + (F_z \circ \mathbf{r}_i)\mathbf{r}_i^*(\mathrm{d}x \wedge \mathrm{d}y))\\ &= \sum_i \iint_{\varphi_i(U_i)} \tilde{\varepsilon}_i((F_x \circ \mathbf{r}_i)\mathbf{r}_i^*(\mathrm{d}y \wedge \mathrm{d}z) + (F_y \circ \mathbf{r}_i)\mathbf{r}_i^*(\mathrm{d}z \wedge \mathrm{d}x) + (F_z \circ \mathbf{r}_i)\mathbf{r}_i^*(\mathrm{d}x \wedge \mathrm{d}y))\\ &= \sum_i \iint_{\varphi_i(U_i)} \tilde{\varepsilon}_i\mathbf{r}_i^*(F_x\mathrm{d}y \wedge \mathrm{d}z + F_y\mathrm{d}z \wedge \mathrm{d}x + F_z\mathrm{d}x \wedge \mathrm{d}y)\end{aligned}$$

Thus, if we let $\psi = F_x\mathrm{d}y \wedge \mathrm{d}z + F_y\mathrm{d}z \wedge \mathrm{d}x + F_z\mathrm{d}x \wedge \mathrm{d}y$, then we can write that

$$\oiint_S \mathrm{d}\mathbf{S} \cdot \mathbf{F} = \sum_i \iint_{\varphi_i(U_i)} \tilde{\varepsilon}_i \mathbf{r}_i^* \psi$$

$$= \sum_i \iint_{\varphi_i(U_i)} \mathbf{r}_i^* (\varepsilon_i' \psi)$$

$$= \sum_i \iint_{\varphi_i(U_i)} S_i^* (\iota^* (\varepsilon_i' \psi))$$

$$= \sum_i \iint_{\varphi_i(U_i)} (\varphi_i^{-1})^* (\iota^* \circ (\varepsilon_i' \psi))$$

$$= \int_S \iota^* \psi$$

$$= \int_{\partial V = S} \psi.$$

Assuming ψ has compact support, then we can use Stokes' theorem to rewrite this integral as

$$\int_V \mathrm{d}\psi.$$

Now, to calculate $\mathrm{d}\psi$, we use the definition of the exterior derivative to see that

$$\mathrm{d}\psi = \mathrm{d}(F_x \mathrm{d}y \wedge \mathrm{d}z + F_y \mathrm{d}z \wedge \mathrm{d}x + F_z \mathrm{d}x \wedge \mathrm{d}y)$$

$$= \mathrm{d}F_x \wedge \mathrm{d}y \wedge \mathrm{d}z + \mathrm{d}F_y \wedge \mathrm{d}z \wedge \mathrm{d}x + \mathrm{d}F_z \wedge \mathrm{d}x \wedge \mathrm{d}y$$

$$= \left(\frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z} \right) \mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z$$

$$= (\mathrm{div}\mathbf{F}) \mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z.$$

Now, evaluating this integral, we get

$$\int_V \mathrm{d}\psi = \int_V (\mathrm{div}\mathbf{F})\mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z$$
$$= \iiint\limits_V (\mathrm{div}\mathbf{F})\mathrm{d}V,$$

establishing the divergence theorem as a special case of Stokes' theorem.

2.4 de Rham Cohomology

The key insight that allows us to construct a form of cohomology out of differential forms is that we have a sequence of spaces $\Omega^k(M)$, with a mapping $\mathrm{d} : \Omega^k(M) \to \Omega^{k+1}(M)$ which satisfies $d^{\circ 2} = 0$. In other words, we have an ideal candidate for a cochain complex in the pair $(\{\Omega^k(M)\}_{i=-\infty}^{\infty}, \mathrm{d})$, (where $\Omega^k(M) = \{0\}$ for $k > n, n < 0$). Thus, in order to construct a cohomology, we need only define our coboundaries and cocycles.

Definition 2.4.1 (Closed forms)**.** A closed differential form α is one such that

$$\mathrm{d}\alpha = 0.$$

Denote the set of closed forms of degree k as $Z^k(M)$.

Definition 2.4.2 (Exact forms)**.** An exact differential form α is one such that there exists a differential form β satisfying

$$\alpha = \mathrm{d}\beta.$$

Denote the set of exact forms of degree k as $B^k(M)$.

Notice that, as expected, $\mathrm{d}^{\circ 2} = 0$, meaning $B^k(M) \subseteq Z^k(M)$.

Definition 2.4.3 (The de Rham cohomology)**.** Let the k^{th} de Rham cohomology group be defined as

$$H^k_{\text{d.R.}}(M) \equiv Z^k(M)/B^k(M).$$

It turns out that Stokes' theorem expresses the duality between differential forms and singular chain complexes in such a way that suggests that this de Rham cohomology is dual to the singular homology. To see this, note that Stokes' theorem essentially says that

$$(c, \mathrm{d}\omega) = (\partial c, \omega)$$

when we define a bilinear form $(\cdot,\cdot) : C_k(M) \times \Omega^k(M)$ by $(c,\omega) = \int_c \omega$. Now, we will see that this bilinear form induces a pairing between $H_r(M)$ and $H^r_{\text{d.R.}}(M)$.

Theorem 2.4.4 (The de Rham theorem)**.** *Let $H_r(M;\mathrm{R})$ be the singular homology of M with coefficients in $\mathbb{R}$, as defined in [Lee11, p. 343] and [Hat00, p. 153]. Then, the map*

$$\Lambda : H_r(M;\mathbb{R}) \times H^r_{\text{d.R.}}(M) \to \mathbb{R}$$

defined by $\Lambda([c],[\omega]) \equiv (c,\omega) = \int_c \omega$ is bilinear, well-defined, and non-degenerate. In other words, $H^r_{\text{d.R.}}(M) \cong (H_r(M))^*$.

Proof. I will only show the first two claims to be true. To see the last proved, cf. [Lee12, Ch. 18]. To prove the first, notice that $(\cdot,\cdot)$ is bilinear and thus, if Λ is well-defined, it will inherit this structure. Thus, we must only show that for any $c \in Z_k(M)$, $b \in B_k(M)$ and $\omega \in Z^k(M)$ and $\beta \in B^k(M)$, that

$$\int_{c+b} (\omega + \beta) = \int_c \omega.$$

Equivalently, we must show that

$$\int_b \omega, \int_b \beta, \text{ and } \int_c \beta = 0.$$

We can show the first two to be true, by noticing that $b \in B_k(M)$ means that it is equal to ∂d for some $d \in C_{k+1}(M)$. Thus, we have by Stokes' theorem, that for any $\alpha \in Z^k(M)$,

$$\int_{\partial d} \alpha = \int_d \mathrm{d}\alpha = \int_d 0 = 0,$$

by definition of $Z^k(M)$. As $B^k(M) \subseteq Z^k(M)$, this means that

$$\int_b \omega = \int_b \beta = 0.$$

Now, we will consider

$$\int_c \beta.$$

As $\beta \in B^k(M)$, $\beta = \mathrm{d}\alpha$ for some $\alpha \in \Omega^{k-1}(M)$. Thus, by Stokes' theorem, we have that

$$\int_c \mathrm{d}\alpha = \int_{\partial c} \alpha = \int_0 \alpha = 0,$$

as $c \in Z_k(M)$, so $\partial c = 0$. □

Example 2.4.5 (de Rham cohomology of a circle)**.** We can view S^1 as $\{z \in \mathbb{C} : |z| = 1\}$. Let $\mathrm{cis} : \mathbb{R} \to S^1$ be the map $\mathrm{cis}(x) = e^{ix}$. cis is a smooth covering map, meaning that it is also a smooth surjective submersion [Lee12, Prop. 4.33]. This means that cis^* is injective [Tu11, Prob. 18.8].

Clearly, as $\dim S^1 = 1$, $\Omega^k(S^1) = 0$ for $k > 1$. As such, it is obvious that every 1-form is closed, and that there are no exact 0-forms. But which forms are exact?

If $\beta \in \Omega^1(S^1)$, is exact, then $\beta = \mathrm{d}f$ for $f \in C^\infty(S^1)$. Then,

$$\begin{aligned}\int_{S^1} \beta &= \int_{S^1} \mathrm{d}f \\ &= \int_{\partial S^1} f \\ &= \int_{\{\}} f = 0.\end{aligned}$$

Now, suppose that $\beta \in \Omega^1(S^1)$ and $\int_{S^1} \beta = 0$. Note that

$$\begin{aligned}\int_{S^1} \beta &= \int_{\mathrm{cis}([0,2\pi])} \beta \\ &= \int_{[0,2\pi]} \mathrm{cis}^* \beta.\end{aligned}$$

Let $f(t) := \int_{\gamma(I)} \beta$ for some β such that $\int_{S^1} \beta = 0$, where $\gamma : I \to S^1$ is an arbitrary curve in S^1 between 1 and $\mathrm{cis}(t)$. Then, up to a reparametrization, $\gamma = \mathrm{cis}$, with $x \in [0, t + 2\pi n]$, $n \in \mathbb{Z}$. As $\int_{S^1} \beta = 0$, this is a well-defined map.

Let $\mathrm{cis}^* \beta = g(t)\mathrm{d}t$. Then, $f(t) = \int_{\mathrm{cis}[0,t]} \beta = \int_{[0,t]} \mathrm{cis}^* \beta = \int_{[0,t]} g(t)\mathrm{d}t = \int_0^t g(t)\mathrm{d}t$. By the fundamental theorem of calculus, $f'(t) = g(t)$, so clearly, $\mathrm{d}f = g(t)\mathrm{d}t$. Now, we know that $g(t)\mathrm{d}t = \mathrm{cis}^* \beta$, so $\mathrm{d}f = \mathrm{cis}^* \beta$. Define $\tilde{f}(p) : S^1 \to \mathbb{R}$ by $\tilde{f}(p) = f(\mathrm{cis}^{-1}(p))$. Clearly by the analysis above $\tilde{f}(p)$ is well defined (and smooth). Furthermore $f = \mathrm{cis}^* \tilde{f}$. Thus, $\mathrm{d}(\mathrm{cis}^* \tilde{f}) = \mathrm{cis}^* \beta$, so $\mathrm{cis}^*(\mathrm{d}\tilde{f}) = \mathrm{cis}^* \beta$. As cis^* is injective, $\mathrm{d}\tilde{f} = \beta$, meaning β is exact.

Now, we see that the map $\beta \to \int_{S^1} \beta$ descends into a well-defined map $H^1(S^1) \to \mathbb{R}$. Furthermore, the map is injective, because if any two forms β_1, β_2 have the same integral, then $\beta_1 - \beta_2$ integrates to 0, meaning it is exact, meaning $[\beta_1] = [\beta_2]$. But does there exist any non-zero cohomology class in $H^1(S^1)$? To answer in the affirmative, we must provide a closed one-form on S^1 with nonzero integral.

Let $\alpha = \frac{(-y)\mathrm{d}x + x\mathrm{d}y}{x^2+y^2} \in \Omega^2(\mathbb{R}^2)$, and $\iota : S^1 \hookrightarrow \mathbb{R}^2\backslash\{0\}$ be the inclusion of the circle into the punctured plane defined by $\iota z = (\mathrm{Re}z, \mathrm{Im}z)$. Clearly, $\mathrm{d}\alpha = 0$ on the punctured plane. Now consider $\iota^*\alpha$. This is closed on S^1 as well. We have that

$$
\begin{aligned}
\int_{S^1} \iota^*\alpha &= \int_{S^1} \iota^* \left(\frac{(-y)\mathrm{d}x + x\mathrm{d}y}{x^2+y^2} \right) \\
&= \int_{[0,2\pi]} \mathrm{cis}^* \iota^* \left(\frac{(-y)\mathrm{d}x + x\mathrm{d}y}{x^2+y^2} \right) \\
&= \int_{[0,2\pi]} \left(\frac{(-y \circ \iota \circ \mathrm{cis})\mathrm{d}(x \circ \iota \circ \mathrm{cis}) + x \circ \iota \circ \mathrm{cis}\mathrm{d}(y \circ \iota \circ \mathrm{cis})}{x \circ \iota \circ \mathrm{cis}^2 + y \circ \iota \circ \mathrm{cis}^2} \right) \\
&= \int_{[0,2\pi]} \frac{-\sin t \mathrm{d} \cos t + \cos t \mathrm{d} \sin t}{\cos^2 t + \sin^2 t} \\
&= \int_{[0,2\pi]} \mathrm{d}t \\
&= 2\pi
\end{aligned}
$$

Thus, we have a form $\iota^*\alpha$ which is closed, inexact, and with nonzero integral on S^1. That means that $\int_{S^1}$ is a surjective map to the reals, because for any $x \in \mathbb{R}$, $\frac{x}{2\pi}\iota^*\alpha$ integrates to x. Thus, we have a one-to-one correspondence between $H^1(S^1)$ and $\mathbb{R}$.

Now, let's consider which 0-forms are closed. We know that constant functions are closed 0-forms, but are they the only closed 0-forms? Suppose $f \in C^\infty(S^1)$ and $\mathrm{d}f = 0$. Then $\mathrm{cis}^*\mathrm{d}f = 0$ as well. We then have that $\mathrm{d}(\mathrm{cis}^* f) = 0$, meaning that $X(f \circ \mathrm{cis}) = 0$ for all $X \in T\mathbb{R}$. As all $X \in T\mathbb{R}$ can be written as $g\frac{\mathrm{d}}{\mathrm{d}t}$, we then have that for all $g \in C^\infty(\mathbb{R})$, for all t, $g(t)\frac{\mathrm{d}}{\mathrm{d}t}f(\mathrm{cis}(t))|_t = 0$, which is clearly only possible if $f \circ \mathrm{cis}$ is a constant map, which in turn means f must be constant, as cis satisfies $\mathrm{cis}(t + \delta) \neq \mathrm{cis}(t)$ if $\delta < 2\pi$. Thus, $H^0(S^1) = \{\text{constant functions}\}/\{\} = \{\text{constant functions}\} \cong \mathbb{R}$.

Chapter 3: Equivariant Cohomology

Often one has to deal with spaces which have some sort of symmetry. These symmetries are usually encoded in terms of group actions. These group actions give us more information about our manifold, which we would like to be able to take into account when we analyze the cohomology of our space, but the standard de Rham cohomology does not do so. As such, we will have to introduce a new form of cohomology, equivariant cohomology, which works in the category of G-manifolds rather than merely smooth manifolds.

We begin this chapter by introducing the necessary theory of Lie groups and Lie algebras, as well as group actions, before finally giving a solid definition of the relevant model of equivariant cohomology we will be using.

3.1 Lie Groups and Lie Algebras

Lie groups are the natural choice of object to describe continuous symmetries. They appear all over in both physics and math, and will be the only type of group we will be considering.

Definition 3.1.1 (Lie group)**.** Let G be a smooth manifold, and $m : G \times G \to G$ and $i : G \to G$ smooth maps. Let G be a group with multiplication m and inversion i. Then (G, m, i) is a Lie group.

We also will make use of the two functions of left and right multiplication, $\ell_g(\cdot) = m(g, \cdot)$ and $r_g = m(\cdot, g)$.

Definition 3.1.2 (Lie algebra)**.** A vector space V endowed with a bilinear product $[\cdot, \cdot] : V \times V \to V$ is a Lie algebra if

(a) $[x, x] = 0$ for all $x \in V$,

(b) $[x,[y,z]]+[y,[z,x]]+[z,[x,y]]=0$ for all $x,y,z\in V$.

If these properties are satisfied, $[\cdot,\cdot]$ is called the Lie bracket, or the commutator.

It follows from this definition that the Lie bracket is anticommutative.

Now, one may wonder why Lie algebras and Lie groups are related. After all, they seem to have unconnected definitions. However, that separation is only skin-deep; every Lie group gives rise to a corresponding Lie algebra.

Definition 3.1.3 (Lie algebra of a Lie group)**.** The Lie algebra $\mathfrak{g}$ of a Lie group G is defined as $\mathfrak{g}\equiv T_eG$, where e is G's identity. $\mathfrak{g}$ is endowed with the Lie bracket $[\cdot,\cdot]:T_eG\times T_eG\to T_eG$ defined by

$$[X,Y](f)=X(\tilde{Y}(f))-Y(\tilde{X}(f)),$$

where given $Z\in T_eG$, $\tilde{Z}$ is the vector field defined by $\tilde{Z}_g=(\ell_g)_*Z$.

We see that $[X, Y] \in T_e(G)$ by considering

$$
\begin{aligned}
[X,Y](fg) &= X(\tilde{Y}(fg)) - Y(\tilde{X}(fg)) \\
&= X(f\tilde{Y}(g) + g\tilde{Y}(f)) - Y(f\tilde{X}(g) + g\tilde{X}(f)) \\
&= f(e)X(\tilde{Y}(g)) + (\tilde{Y}(g))(e)X(f) + g(e)X(\tilde{Y}(f)) + (\tilde{Y}(f))(e)X(g) \\
&\quad - f(e)Y(\tilde{X}(g)) - (\tilde{X}(g))(e)Y(f) - g(e)Y(\tilde{X}(f)) - (\tilde{X}(f))(e)Y(g) \\
&= f(e)(X(\tilde{Y}(g)) - Y(\tilde{X}(g))) + g(e)(X(\tilde{Y}(f)) - Y(\tilde{X}(f))) \\
&\quad + (\tilde{Y}(g))(e)X(f) + (\tilde{Y}(f))(e)X(g) - (\tilde{X}(g))(e)Y(f) - (\tilde{X}(f))(e)Y(g) \\
&= f(e)(X(\tilde{Y}(g)) - Y(\tilde{X}(g))) + g(e)(X(\tilde{Y}(f)) - Y(\tilde{X}(f))) \\
&\quad + Y(g)X(f) + Y(f)X(g) - X(g)Y(f) - X(f)Y(g) \\
&= f(e)(X(\tilde{Y}(g)) - Y(\tilde{X}(g))) + g(e)(X(\tilde{Y}(f)) - Y(\tilde{X}(f))) \\
&= f(e)[X,Y](g) + g(e)[X,Y](f),
\end{aligned}
$$

meaning that it is a derivation at e. We used that $(\tilde{Z}(f))(e) = (\tilde{Z}_e(f)) = Z(f)$ to get to the penultimate equality. It is a straightforward, if somewhat tedious, calculation to show that this Lie bracket satisfies the Jacobi identity [Lee12, Prop. 8.28].

Now, as $\mathfrak{g}$ is a vector space, and G is a group, one may consider the representations of G on $\mathfrak{g}$. Luckily for us, there is one natural choice.

Definition 3.1.4 (The adjoint representation)**.** Let $c_g(h) = \ell_g r_g^{-1}(h) = ghg^{-1}$ be the conjugation map. Then we have that $\mathrm{Ad}_g \equiv (c_g)_*|_{T_eG} : T_eG \to T_eG$, the **adjoint representation**, is the restriction of the pushforward of the conjugation to the Lie algebra.

This is referred to as a representation because $\mathrm{Ad} : g \to \mathrm{Ad}_g$ is a homomorphism from G to $\mathrm{Aut}\mathfrak{g}$, or in other words, a representation of G.

There is also a map from $\mathfrak{g} \to G$, called the exponential map, which can be quite useful.

Definition 3.1.5 (The exponential map)**.** Let $\gamma_X : \mathbb{R} \to G$ be the unique (we will not prove this) smooth homomorphism such that $\gamma'_X(0) = X$. Then exponential map $\exp : \mathfrak{g} \to G$ is defined as $\exp(X) = \gamma_X(1)$. $\exp(X)$ can also be written as e^X.

Theorem 3.1.6.

$$\gamma_X(t) = \exp(tX).$$

Proof. To show that $\exp(tX)$ is a smooth homomorphism, we consider the map $\lambda : \tau \to \gamma_X(t\tau)$. Clearly, this is a smooth homomorphism. Now, by the fact that the pushforward of a scalar multiplication $\mu^t : x \to tx$ is in turn defined by $\mu^t_*(\partial_x|_{x_0}) = t\partial_x|_{(tx_0)}$, and that pushforwards are linear maps, we see that $\lambda'(0) = \lambda_*(\partial_\tau|_0) = (\gamma_X)_*(\mu^t_*(\partial_\tau|_0) = (\gamma_X)_*(t\partial_\tau|_0) = t(\gamma_X)_*(\partial_\tau|_0) = tX$.

Thus, we see that $\lambda(\tau) = \gamma_X(t\tau) = \gamma_{tX}(\tau)$, which in turn means that $\exp(tX) = \gamma_{tX}(1) = \gamma_X(t)$.

□

Theorem 3.1.7. *Given some map $F : G \to M$, we have that*

$$F_*(X) = (F \circ \gamma_X)'(0).$$

We also have that $X(f) = (f \circ \gamma_X)'(0)$, with the derivative taken as a standard derivative rather than a velocity vector in the latter case.

Proof. This follows from recalling that a curve's velocity at t is the pushforward of $\partial_t|_t$.

Then, we see that

$$
\begin{aligned}
(F \circ \gamma_X)'(0) &= F_* \circ (\gamma_X)_*(\partial_t|_0) \\
&= F_*((\gamma_X)_*(\partial_t|_0)) \\
&= F_*(\gamma_X'(0)) \\
&= F_*(X).
\end{aligned}
$$

To show that $X(f) = (f \circ \gamma_X)'(0)$, taken in the sense of standard limit-based derivatives rather than pushforwards, notice that

$$
\begin{aligned}
X(f) &= (\gamma_X'(0))(f) \\
&= ((\gamma_X)_*(\partial_t|_{t=0}))(f) \\
&= \partial_t|_{t=0} f(\gamma_X(t)) = (f \circ \gamma_X)'(0).
\end{aligned}
$$

□

This proof in fact holds more generally. Given a map $F : M \to N$, and any curve $\Gamma_V : \mathbb{R} \to M$ with $V \in T_{\Gamma_V(0)}M$ such that $\Gamma_V'(0) = V$, $F_*(V) = (F \circ \Gamma_V)'(0)$. Furthermore, if $(g \cdot \gamma_X)(t) = g \cdot \gamma_X(t)$, we have that $\tilde{X}_g(f) = (f \circ (g \cdot \gamma_X))'(0)$, as $\tilde{X}_g = (\ell_g)_* X$, so

$$
\begin{aligned}
\tilde{X}_g(f) &= X(f \circ \ell_g) \\
&= (f \circ \ell_g \circ \gamma_X)'(0) \\
&= (f \circ (g \cdot \gamma_X))'(0),
\end{aligned}
$$

once again as actual derivatives, not pushforwards. Thus, $\tilde{X}(f) \in C^\infty(G)$ sends $g \in M$ to

$(f \circ (g \cdot \gamma_X))'(0)$.

This allows us to rewrite the Lie bracket in terms of a derivative of a pushforward.

Theorem 3.1.8. *Given $X, Y \in \mathfrak{g}$,*

$$[X,Y] = \lim_{t\to 0} \frac{(r_{e^{-tX}})_*(\ell_{e^{tX}})_* Y - Y}{t}$$

$$= \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} ((r_{e^{-tX}})_* \tilde{Y})_e.$$

Proof. We begin with the definition of the Lie bracket of $\mathfrak{g}$:

$$[X,Y] = X \circ \tilde{Y} - Y \circ \tilde{X}.$$

Acting on some function f, that gives us

$$[X,Y](f) = X(\tilde{Y}(f)) - Y(\tilde{X}(f))$$

$$= ((\tilde{Y}(f)) \circ \gamma_X)'(0) - ((\tilde{X}(f)) \circ \gamma_Y)'(0)$$

$$= \frac{\partial^2}{\partial t \partial \tau} (f(\gamma_Y(t) \cdot \gamma_X(\tau)) - f(\gamma_X(t) \cdot \gamma_Y(\tau))\Big|_{t=\tau=0}$$

$$= \frac{\partial^2}{\partial t \partial \tau} \left(f(e^{tY} \cdot e^{\tau X}) - f(e^{tX} \cdot e^{\tau Y})\right)\Big|_{t=\tau=0}$$

Now let's consider

$$\left(\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} ((r_{e^{-tX}})_* \tilde{Y})_e\right)(f) = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \tilde{Y}_{e^{tX}}(f \circ r_{e^{-tX}})$$

$$= \frac{\mathrm{d}}{\mathrm{d}t} \left((f \circ r_{e^{-tX}} \circ (e^{tX} \cdot \gamma_Y))'(0)\right)\Big|_{t=0}$$

$$= \frac{\partial^2}{\partial t \partial \tau} (f(e^{tX} \cdot e^{\tau Y} \cdot e^{-tX}))\Big|_{t=\tau=0}.$$

Let $F(t_1, \tau, t_2) = f(e^{t_1 X} \cdot e^{\tau Y} \cdot e^{-t_2 X})$. By the chain rule, $\frac{\partial}{\partial t} F(t, \tau, t)|_{t=0} = \frac{\partial}{\partial t_1} F(0, \tau, 0) +$

$\frac{\partial}{\partial t_2}F(0,\tau,0)$. Thus, we have that

$$\frac{\partial^2}{\partial t \partial \tau}(f(e^{tX}\cdot e^{\tau Y}\cdot e^{-tX}))\Big|_{t=\tau=0} = \frac{\partial}{\partial \tau}\Big|_{\tau=0}\left(\frac{\partial}{\partial t}\Big|_{t=0}(f(e^{tX}\cdot e^{\tau Y}))\right.$$
$$\left. + \frac{\partial(-t)}{\partial t}\frac{\partial}{\partial(-t)}\Big|_{(-t)=0}(f(e^{\tau Y}\cdot e^{-tX}))\right)$$
$$= \frac{\partial^2}{\partial t \partial \tau}\left(f(e^{tY}\cdot e^{\tau X}) - f(e^{tX}\cdot e^{\tau Y})\right)\Big|_{t=\tau=0} = [X,Y](f),$$

where we relabeled $-t$ as t to get to the last line. Thus, as the two vectors have equal action on any functions, they themselves in turn must be equal. □

We also have that the exponential map is a natural transformation between Lie algebras and Lie groups.

Lemma 3.1.9. *If $f : G \to H$ is a smooth homomorphism of Lie groups, then $f_* : \mathfrak{g} \to \mathfrak{h}$ satisfies $f(\exp(X)) = \exp(f_*(X))$ for all $X \in \mathfrak{g}$.*

Proof. Note that $f(\exp(X)) = f(\gamma_X(1))$. Consider $f(\gamma_X(t))$. Clearly $f \circ \gamma_X : \mathbb{R} \to H$ is a smooth homomorphism. Furthermore, $(f \circ \gamma_X)'(0) = (f \circ \gamma_X)_*(\partial_t|_0) = f_*(X)$, so it is the unique smooth homomorphism $\gamma_{f_*X} : \mathbb{R} \to N$ such that $\gamma'_{f_*X}(0) = f_*X$. Thus, $\exp(f_*X) = \gamma_{f_*X}(1) = f(\gamma_X(1)) = f(\exp(X))$. □

Corollary 3.1.9.1. *For all $g \in G$, $X \in \mathfrak{g}$,*

$$\exp(\mathrm{Ad}_g(X)) = g\exp(X)g^{-1}.$$

We are now equipped to show that the derivative of the adjoint representation is the commutator.

Theorem 3.1.10. *Let $\mathrm{Ad} : G \to \mathrm{Aut}(\mathfrak{g})$ be the function defined by $\mathrm{Ad}(g) = \mathrm{Ad}_g$ Then $\mathrm{Ad}_* : \mathfrak{g} \to T_I\mathrm{Aut}(\mathfrak{g}) \cong \mathrm{End}(\mathfrak{g})$ satisfies $(\mathrm{Ad}_*(X))(Y) = [X,Y]$.*

We first prove a technical lemma that helps simplify handling the pushforwards of maps to $\mathrm{Aut}(\mathfrak{g})$.

Lemma 3.1.11. *Let $F : G \to \mathrm{Aut}(\mathfrak{g})$ be a smooth Lie group homomorphism, and let $F[Y] : G \to \mathfrak{g}$ be defined by $(F[Y])(g) = (F(g))(Y)$. Then, we have that $(F_*(X))(Y) = (F[Y])_*(X)$, with $F_* : \mathfrak{g} \to \mathrm{End}(\mathfrak{g})$ and $(F[Y])_* : \mathfrak{g} \to \mathfrak{g}$.*

Proof.

$$\begin{aligned} F_*(X) &= \frac{\mathrm{d}}{\mathrm{d}t} F(e^{tX})\Big|_{t=0} \\ (F_*(X))(Y) &= \left(\frac{\mathrm{d}}{\mathrm{d}t} F(e^{tX})\Big|_{t=0} \right)(Y) \\ &= \frac{\mathrm{d}}{\mathrm{d}t}((F(e^{tX}))(Y))|_{t=0}. \end{aligned}$$

where, as $\mathrm{Aut}(\mathfrak{g})$ and $\mathfrak{g}$ are finite dimensional vector spaces and F is smooth, we can treat the right hand sides as actual derivatives.

Now, consider $((F[Y])_*(X))$. We have that

$$\begin{aligned} ((F[Y])_*(X)) &= \frac{\mathrm{d}}{\mathrm{d}t} F[Y](e^{tX})\Big|_{t=0} \\ &= \frac{\mathrm{d}}{\mathrm{d}t} (F(e^{tX}))(Y)\Big|_{t=0} \\ &= (F_*(X))(Y). \end{aligned}$$

□

Proof of Thm. (3.1.10). First, we need to explain why $T\mathrm{Aut}(\mathfrak{g}) \cong \mathrm{End}(\mathfrak{g})$. As $\mathrm{Aut}(\mathfrak{g}) = \{M \in \mathrm{End}(\mathfrak{g}) \ : \ \det M \neq 0\}$, and we know that the determinant is continuous (as it is a polynomial function of matrix entries, which are linear functions $\mathrm{End}(\mathfrak{g}) \to \mathbb{R}$) by the fact that the preimage of an open set under a continuous map is open, we see that $\mathrm{Aut}(\mathfrak{g})$ is an open submanifold of $\mathrm{End}(\mathfrak{g})$. As we know that given an open submanifold $U \subseteq M$ that

$T_pU \cong T_pM$, we see that $T_I(\mathrm{Aut}(\mathfrak{g})) \cong T_I(\mathrm{End}(\mathfrak{g})) \cong \mathrm{End}(\mathfrak{g})$, as the tangent space to a finite-dimensional vector space V is isomorphic to V.

Now, to evaluate the pushforward of Ad at the identity, we make use of Lemma (3.1.11) to say that, if $\mathrm{A}[Y] : G \to \mathfrak{g}$ is defined by $\mathrm{A}[Y](g) = \mathrm{Ad}_g(Y)$

$$\begin{aligned}
(\mathrm{Ad}_*(X))(Y) &= (\mathrm{A}[Y])_*(X) \\
&= \frac{\mathrm{d}}{\mathrm{d}t} \mathrm{Ad}_{e^{tX}}(Y)\Big|_{t=0} \\
&= \frac{\mathrm{d}}{\mathrm{d}t} \left(\ell_{e^{tX}} \circ r_{e^{-tX}}\right)_* Y\Big|_{t=0}.
\end{aligned}$$

Now, as this is a proper derivative, and thus an element of $\mathfrak{g}$, we can consider its action on a function $f \in C^\infty(G)$. We have that

$$\begin{aligned}
((\mathrm{Ad}_*(X))(Y))(f) &= \left(\frac{\mathrm{d}}{\mathrm{d}t} \left(\ell_{e^{tX}} \circ r_{e^{-tX}}\right)_* (Y)\Big|_{t=0}\right)(f) \\
&= \frac{\mathrm{d}}{\mathrm{d}t} \left(\left(\left(\ell_{e^{tX}} \circ r_{e^{-tX}}\right)_* Y\right)(f)\right)\Big|_{t=0} \\
&= \frac{\mathrm{d}}{\mathrm{d}t} Y(f \circ \ell_{e^{tX}} \circ r_{e^{-tX}})\Big|_{t=0} \\
&= \frac{\partial^2}{\partial t \partial \tau} f(e^{tX} e^{\tau Y} e^{-tX})\Big|_{t=\tau=0},
\end{aligned}$$

which, as we've shown above in Thm. (3.1.8) is just $[X, Y](f)$. Thus, we have that $(\mathrm{Ad}_*(X))(Y) = [X, Y]$, so $\mathrm{Ad}_* : \mathfrak{g} \to \mathrm{End}(\mathfrak{g})$ sends $X \to [X, \cdot]$. □

Now, we come to a remarkable result relating three of the most important operations in Lie groups and Lie algebras: the exponential map, the Lie bracket, and the adjoint representation.

Theorem 3.1.12. *Given* $X \in \mathfrak{g}$,

$$\mathrm{Ad}_{e^X}(\cdot) = \mathrm{Exp}([X, \cdot]),$$

where $\mathrm{Exp} : \mathrm{End}(\mathfrak{g}) \to \mathrm{Aut}(\mathfrak{g})$ *is defined by*

$$\mathrm{Exp}(f) = \sum_{k=0}^{\infty} \frac{(f)^{\circ k}}{k!}.$$

Proof. Note that $\mathrm{End}(\mathfrak{g})$ is a Lie algebra with bracket $[f, g] = f \circ g - g \circ f$. In this context, it is fairly straightforward to show that Exp is defined for all of $\mathrm{End}(\mathfrak{g})$ [Hal15, Prop 2.1], and is the exponential map for this Lie algebra, mapping it into $\mathrm{Aut}(\mathfrak{g})$, the Lie group in this context [Hal15, Prop 2.3].

Thus, we see that, as $\mathrm{Ad}_*(X) = [X, \cdot]$ by Thm. (3.1.10), the result is a straighforward application of Lemma (3.1.9). ☐

Note that while we haven't defined a norm for $\mathfrak{g}$ or $\mathrm{End}(\mathfrak{g})$, if G (and thus $\mathfrak{g}$ and $\mathrm{End}(\mathfrak{g})$) is finite dimensional, which it is for the concerns of this paper, all norms are equivalent, so convergence makes sense.

Corollary 3.1.12.1. *Given* $X, Y \in \mathfrak{g}$ *such that* $[X, Y] = 0$, *then* $\mathrm{Ad}_{e^X}(Y) = Y$.

Finally, note once again that the Lie bracket is intrinsically tied up with commutativity, in that the Lie algebra of an abelian Lie group has trivial bracket.

Theorem 3.1.13. *If* G *is an Abelian Lie group, then for any* $X, Y \in \mathfrak{g}$, $[X, Y] = 0$.

Proof. Notice that if G is Abelian, $c_g = \mathrm{id}_G$ for all $g \in G$. Thus, the differential, by which the adjoint representation is defined, must be the constant map which takes $g \to \mathrm{id}_{\mathfrak{g}}$. Thus, the pushforward of Ad must be the zero map. Thus, as $\mathrm{Ad}_*(X) = [X, \cdot]$, we see that $[X, Y] = 0$ for all $X, Y \in \mathfrak{g}$. ☐

3.2 Group Actions

Lie groups and Lie algebras are a fascinating subject in their own right, but for our purposes, we care about them insofar as they are related to symmetries, and symmetries are properties of a space, that is, a smooth manifold. Thus, we need to have a way to relate them to manifolds. This is where the language of group actions comes in.

Definition 3.2.1 (Group actions, cf. [Lee12],[Tu20])**.** Let G be a group, and M a smooth manifold. A left group action is a homomorphism $\mu : G \to \mathrm{Diff}(M)$, that is, a map μ that sends g to $\mu(g) : M \to M$ in such a way that $\mu(g_1 g_2) = \mu(g_1) \circ \mu(g_2)$. We will also denote $\mu(g)$ by μ_g or, given $p \in M$, by $g \cdot p$. A right group action is an antihomomorphism $\mu : G \to \mathrm{Diff}(M)$, i.e., a map such that $\mu(g_1) \circ \mu(g_2) = \mu(g_2 g_1)$. Right group actions may also be denoted by μ_g or, given $p \in M$, by $p \cdot g$

One often calls a smooth manifold M with a group action $\mu : G \to \mathrm{Diff}(M)$ a smooth G-manifold.

We will work with left actions mostly for the rest of this thesis except when discussing principal bundles, for which right actions are more commonly used. Note that a left action l can be made into a right action by letting $r(g) \equiv l(g^{-1})$, so that $r(gh) = l((gh)^{-1}) = l(h^{-1}g^{-1}) = l(h^{-1}) \circ l(g^{-1}) = r(h) \circ r(g)$, and vice versa.

3.2.1 Types of Group Actions

Now we will introduce some specific classes of group actions we make use of later.

Definition 3.2.2 (Free group actions)**.** A group action μ on M is free if for all $p \in M$, $g \in G\backslash\{I\}$, $\mu_g(p) \neq p$.

Definition 3.2.3 (Transitive group actions)**.** A group action μ on M is transitive if for any points $p_1, p_2 \in M$ there exists $g \in G$ such that $\mu_g(p_1) = p_2$.

One important property a group action is whether or not it leaves some points fixed. If it does, we often care about those specific points.

Definition 3.2.4 (Fixed points)**.** Given a G-manifold M, $p \in M$ is a fixed point of M if for all $g \in G$, $\mu_g p = p$. The set of fixed points is denoted M^G.

3.2.2 Fundamental Vector Fields

Fundamental vector fields characterize the infinitesimal behaviour of our group action at points. For our purposes, though, they will be primarily used in order to "include" the Lie algebra $\mathfrak{g}$ into the space of vector fields on M.

Definition 3.2.5 (The fundamental vector field)**.** Let the map $\mu(p) : G \to M$ be defined as $\mu(p)(g) = \mu_g(p)$. The fundamental vector field $X^{\#}$ associated to $X \in \mathfrak{g} = T_eG$ is defined as

$$(X^{\#})_p = (\mu(p))_*(X).$$

Due to fundamental vector fields, we can speak of interior multiplication by an element of $X \in \mathfrak{g}$, in that $\iota_X \equiv \iota_{X^{\#}}$.

Furthermore, we have that the map $\# : X \to X^{\#}$ is G-equivariant.

Theorem 3.2.6. *Let $g \in G$, $X \in \mathfrak{g}$, and M be a smooth G-manifold. Then,*

$$(\mathrm{Ad}_g X)^{\#} = \mu(g)_* X^{\#}.$$

Proof. Let's start with the right-hand side:

$$(\mu(g)_* X^{\#})_p = \mu(g)_* \circ \mu(g^{-1} \cdot p)_* X.$$

$\mu(g) \circ \mu(g^{-1}p)$ sends h to $(ghg^{-1}) \cdot p = c_g(h) \cdot p = \mu(p) \circ c_g(h)$, so $\mu(g) \circ \mu(g^{-1} \cdot p) = \mu(p) \circ c_g$ Now, recalling that $\mathrm{Ad}_g = (c_g)_*$, and using the chain rule for pushforwards, we see that

$$((\mu(g))_* X^{\#})_p = (\mu(p) \circ c_g)_* X = \mu(p)_*((c_g)_* X) = \mu(p)_* \mathrm{Ad}_g(X) = (\mathrm{Ad}_g X)^{\#}_p.$$

□

3.2.3 Quotient Manifolds

When we have a smooth G-manifold M, we can consider the quotient space M/G consisting of the set of G-orbits of points. This is a topological space, but in some cases we can impose more structure.

Theorem 3.2.7. *Let G be a compact manifold acting smoothly and freely on a smooth manifold M. Then M/G is also a smooth manifold with dimension $\dim M - \dim G$. Furthermore, the canonical projection $\pi : M \to M/G$ is a smooth submersion.*

Proof. cf. [Lee12, Thm. 21.10]. □

This theorem starts to help us build intuition as to what we want out of our new form of cohomology. When G acts freely on M, it would be nice for our cohomology to simply take the form of $H^\bullet(M/G)$. However, we cannot just have it be $H^\bullet(M/G)$ in all cases. Two reasons are illustrative of the inadequacy of such a model. First, in the general case, M/G is not always well-behaved enough to work with. M/G is only guaranteed to be a manifold for free actions. Second, even if M/G is a manifold for a non-free action, it may have trivial cohomology useless for us. Tu gives us the example of a sphere being rotated around an axis. That quotient manifold is a closed interval, which has trivial cohomology, thus stripping out any information we would find interesting.

3.3 The Cartan Model

Now we will introduce the model for equivariant cohomology we will use for the rest of this paper, the Cartan model. It is worth noting that there are other models for equivariant cohomology, and that all of the models only correspond in the case when G is a compact, connected Lie group.

Definition 3.3.1 (Symmetric powers)**.** The k-th symmetric power of a vector space V, denoted $S^k(V)$, is defined as the quotient of $\times_1^k V$ by the subspace spanned by the union of the following 3 sets:

$$\mathcal{S}_1 \equiv \{(v_1, v_2, \ldots, v_j, \ldots, v_i, \ldots, v_k) | v_1, \cdots, v_k \in V, 1 \leqslant i, j \leqslant k\}$$

$$\mathcal{S}_2 \equiv \{(v_1, v_2, \ldots, av_m, \ldots, v_k) - a(v_1, v_2, \ldots, v_m, \ldots, v_k) | v_1, \cdots, v_k \in V, 1 \leqslant m \leqslant k\}$$

$$\mathcal{S}_3 \equiv \{((v_1, v_2, \ldots, v_m + w_m, \ldots, v_k) - (v_1, v_2, \ldots, v_m, \ldots, v_k)$$

$$- (0, \ldots, w_m, \ldots, 0)) | v_1, \cdots, v_k, w_m \in V, 1 \leqslant m \leqslant k\}.$$

Definition 3.3.2 (The symmetric algebra)**.** Given a vector space V, the symmetric algebra over V, $S(V)$ is defined as

$$\bigoplus_{k=0}^{\infty} S^k(V)$$

with a product $\vee : S^k(V) \times S^m(V) \to S^{k+v}(V)$ defined by

$$[(u_1, \ldots, u_k)] \vee [(w_1, \ldots, w_m)] = [(u_1, \ldots, u_k, w_1, \ldots, w_m)].$$

As such, for $v_1 \neq v_2 \neq \cdots \neq v_m$, we denote $[(v_1, v_2, \cdots, v_m)]$ by $v_1 \vee v_2 \vee \cdots \vee v_m$, or simply $v_1 v_2 \cdots v_m$. If some vectors are equal, we can consolidate them like so: $v_1^{p_1} v_2^{p_2} \cdots v_m^{p_m}$, where p_k is the number of times v_k is multiplied.

Theorem 3.3.3. *Given any basis $\{v_i\}_1^n$ for a vector space V, $S(V) \cong \mathbb{R}[v_1, \ldots, v_n]$, with $S(V)$'s product corresponding to polynomial multiplication, and vice versa.*

Definition 3.3.4 (Polynomial maps on a vector space)**.** Given a vector space V with dual basis $\omega^1, \omega^2, \ldots, \omega^n$ and an element $f = \sum_I a_I (\omega^1)^{i_1} \cdots (\omega^n)^{i_n}$ (summing over all multi-indices) of $\mathbb{R}[V^*] \equiv \mathbb{R}[\omega^1, \ldots, \omega^n]$, a polynomial map on V $p[f] : V \to \mathbb{R}$ is the map defined by

$$p[f](v) = \sum_I a_I(\omega^1(v))^{i_1} \cdots (\omega^n(v))^{i_n}.$$

Note that a polynomial map on $\mathfrak{g}$ $p[\alpha]$ is in one-to-one correspondence with an element of $S(\mathfrak{g}^*)$, and that it (and thus α as well) is endowed with a g-action in the following way:

$$g \cdot p[\alpha](X) = p[\alpha](\mathrm{Ad}_{g^{-1}}(X)).$$

Now, consider the space $S(\mathfrak{g}^*) \otimes \Omega(M)$, which consists of polynomial maps $\beta : \mathfrak{g} \to \Omega(M)$. $\Omega(M)$ has a natural G-action of $g \cdot \omega = \mu^*_{g^{-1}}\omega$, the pullback of the inverse G-action on M (we need it to be $\mu^*_{g^{-1}}$ because $\mu^*_{(gh)^{-1}} = \mu^*_{h^{-1}g^{-1}} = \mu^*_{g^{-1}}\mu^*_{h^{-1}}$. Alternatively, this follows from stipulating that for $\omega \in \Omega^1(M)$, $v \in T(M)$, $(g \cdot \omega)(g \cdot v) = \omega(v)$ and extending to the whole exterior algebra). Thus, we can speak of $S(\mathfrak{g}^*) \otimes \Omega(M)$ as a G-space with action

$$g \cdot (\alpha \otimes \omega) = (g \cdot \alpha) \otimes (g \cdot \omega),$$

or equivalently, for any $X \in \mathfrak{g}$,

$$(g \cdot \beta)(X) \equiv g \cdot \beta(\mathrm{Ad}_{g^{-1}}(X)) = \mu^*_{g^{-1}}\beta(\mathrm{Ad}_{g^{-1}}(X)).$$

The space of all invariant forms γ, which satisfy $g \cdot \gamma = \gamma$ for all $g \in G$, is denoted $(S(\mathfrak{g}^*) \otimes \Omega(M))^G$, or $\Omega_G(M)$. Elements of $\Omega_G(M)$ can be viewed as G-equivariant polynomial maps $p : \mathfrak{g} \to \Omega(M)$, i.e. polynomial maps such that $p(g \cdot X) = g \cdot p(X)$, as we know that for an invariant form β, $g \cdot \beta(\mathrm{Ad}_{g^{-1}}(X)) = \beta(X)$ for all $g \in G$, so

$$\beta(\mathrm{Ad}_g(X)) = g \cdot \beta(\mathrm{Ad}_{g^{-1}}(\mathrm{Ad}_g(X))) = g \cdot \beta(X).$$

Now, let's consider the Cartan differential D :

$$(D\alpha)(X) \equiv \mathrm{d}(\alpha(X)) - \iota_X(\alpha(X)).$$

There are two things to note about this: $D^2 \neq 0$, but when we restrict to the invariant forms, $\Omega_G(M)$, it does square to 0 (we will prove this momentarily). To see that we even can restrict D to $\Omega_G(M)$, note that

$$\begin{aligned} D\alpha(\mathrm{Ad}_g(X)) &= \mathrm{d}(\alpha(\mathrm{Ad}_g(X))) - \iota_{\mathrm{Ad}_g(X)}(\alpha(\mathrm{Ad}_g(X))) \\ &= \mathrm{d}(\mu^*_{g^{-1}}\alpha(X)) - \iota_{\mathrm{Ad}_g(X)}(\mu^*_{g^{-1}}\alpha(X)) \\ &= \mu^*_{g^{-1}}\left(\mathrm{d}(\alpha(X)) - \iota_X\alpha(X)\right) \\ &= \mu^*_{g^{-1}}((D\alpha)(X)) \end{aligned}$$

where to get from the second to the third line we used Thm. (2.3.3), (3.2.6), and that $\iota_X(f^*\omega) = f^*(\iota_{f_*X}\omega)$.

Second, note that D is not compatible with the graded structure of $\Omega(M)$, in the sense that if $\alpha : \mathfrak{g} \to \Omega^k(M)$, $D\alpha$ doesn't send $\mathfrak{g}$ to $\Omega^{k+1}(M)$, so the space of invariant forms must have a more nontrivial grading, which happens to be the following:

$$\Omega^k_G(M) \equiv \bigoplus_{2a+b=k} \left(S^a(\mathfrak{g}^*) \otimes \Omega^b(M)\right)^G.$$

We will demonstrate that this grading is compatible with D momentarily as well.

To show this is a proper cochain complex, we start to examine this with a definite basis of $\mathfrak{g}$, $\{X_i\}_{i=1}^k$ and corresponding dual basis $\{u^i\}_{i=1}^k$ such that $u^i(X_j) = \delta^i_j$. Note that $X = \sum_i u^i(X)X_i$. Thus, $\iota_X = \iota_{\sum_i u^i(X)X_i} = \sum_i u^i(X)\iota_{X_i}$, which shows us how $X \to \iota_X\alpha(X)$ has the same degree as $X \to \mathrm{d}(\alpha(X))$; while the degree as a differential form decrements, the degree as a polynomial goes up by one, meaning the total degree as an equivariant form increments for both terms. With this in mind, we can finally show that $\Omega_G(M)$ is a cochain complex.

Theorem 3.3.5.

$$(\Omega_G^\bullet(M), D)$$

forms a cochain complex.

Proof. We have already shown above how D sends $\Omega_G^k(M)$ to $\Omega_G^{k+1}(M)$. To finish this proof we need to show that $D^2 = 0$ on $\Omega_G(M)$. To do this, let's expand it out:

$$\begin{aligned}
(D^2\alpha)(X) &= D(D\alpha)(X) \\
&= \mathrm{d}((D\alpha)(X)) - \iota_X((D\alpha)(X)) \\
&= \mathrm{d}(\mathrm{d}(\alpha(X)) - \iota_X(\alpha(X))) - \iota_X(\mathrm{d}(\alpha(X)) - \iota_X(\iota_X(\alpha(X)))) \\
&= -(\mathrm{d}(\iota_X(\alpha(X))) + \iota_X(\mathrm{d}(\alpha(X)))) \\
&= -\mathcal{L}_{X^\#}\alpha(X) = 0
\end{aligned}$$

Where to get from the third to fourth lines we used that $\iota_x^2 = \mathrm{d}^2 = 0$, and to get from the fourth to the fifth, we used Cartan's formula for the Lie derivative. To get that the Lie derivative of $\alpha(X)$ along $X^\#$ was 0, we used that

$$\begin{aligned}
(\mathcal{L}_{X^\#}\alpha(X)) &= \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \mu^*_{e^{tX}}\alpha(X) \\
&= \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \alpha(\mathrm{Ad}_{e^{-tX}}X).
\end{aligned}$$

By Cor. (3.1.12.1), we know that

$$\mathrm{Ad}_{e^{-tX}}(X) = X,$$

which is t-invariant, meaning that $\mathcal{L}_{X^\#}\alpha(X) = 0$, which in turn implies $D^2 = 0$.

$\square$

It is also apparent that if α and β are equivariant forms, the form $\alpha \wedge \beta$ defined by $(\alpha \wedge \beta)(X) = \alpha(X) \wedge \beta(X)$ is also equivariant, as

$$\begin{aligned}(\alpha \wedge \beta)(\mathrm{Ad}_g X) &= \alpha(\mathrm{Ad}_g X) \wedge \beta(\mathrm{Ad}_g X)\\ &= (\mu^*_{g^{-1}}\alpha(X)) \wedge (\mu^*_{g^{-1}}\beta(X))\\ &= \mu^*_{g^{-1}}(\alpha(X) \wedge \beta(X))\\ &= (\mu^*_{g^{-1}}(\alpha \wedge \beta))(X).\end{aligned}$$

If $\alpha = \sum_I u^I \omega_I$ and $\beta = \sum_J u^J \varpi_J$, then $\alpha \wedge \beta = \sum_{I,J} u^I u^J \omega_I \wedge \varpi_J$, where I and J are multiindices, and $\{u^i\}_i$ is a basis of $\mathfrak{g}^*$.

Now, as we have created a cochain complex, we can create a cohomology in the standard way: Let $Z^k_G = \{\alpha \in \Omega^k_G(M) : D\alpha = 0\}$ and $B^k_G(M) = \{\beta \in \Omega^k_G(M) : \beta = D\gamma, \gamma \in \Omega^{k+1}_G(M)\}$ be the spaces of equivariant cocycles and coboundaries respectively. Note that $\beta \in Z^k_G(M)$ doesn't mean $\beta(X) \in Z^k(M)$, or even $\beta(X) \in Z^\bullet(M)$, nor does $\beta \in B^k_G(M)$ mean that $\beta(X) \in B^\bullet(M)$. In fact, this is false quite often. Irregardless, one can define the k^{th} Cartan equivariant cohomology group as $H^k_G(M) := Z^k_G(M)/B^k_G(M)$.

Note that $H^\bullet_G(M)$ becomes a ring under the multiplication $[\alpha] \wedge [\beta] = [\alpha \wedge \beta]$, which is well-defined because if $\alpha \in Z^{k_1}_G(M)$ and $\beta = D\gamma \in B^{k_2}_G(M)$, $\alpha \wedge (D\gamma) = (-1)^{k_1} D(\alpha \wedge \gamma) + (-1)^{k_1}(D\alpha) \wedge \gamma = D((-1)^{k_1}\alpha \wedge \gamma) \in B^{k_1+k_2}_G(M)$.

3.3.1 Circle and Torus Actions

Now let us consider the prime examples of the Cartan model we will see in the next section: circle and torus actions.

Notice, that as both those are Abelian, that we can simplify $\Omega_G(M)$ considerably, as $\mathrm{Ad} : G \to \{\mathrm{id}_{\mathfrak{g}}\}$, so for any $\alpha \in \Omega_G(M)$

$$(g \cdot \alpha)(X) = \mu^*_{g^{-1}}\alpha(\mathrm{Ad}_{g^{-1}}(X)) = \mu^*_{g^{-1}}\alpha(X),$$

so for α to be invariant, it need merely be in $S(\mathfrak{g}^*) \otimes \Omega(M)^G$, or, given dual basis $\{u^i\}_{i=1}^k$, $\Omega(M)^G[u^1, u^2, \ldots, u^k]$, i.e. an element of the space of G-invariant-form-valued polynomial maps on $\mathfrak{g}$. For a circle S^1, there is only one element of the dual basis as the Lie algebra is isomorphic to $\mathbb{R}$, so we have $\Omega^{S^1}(M)[u]$. Note that if we have a basis $\{X_i\}$ with dual basis $\{u^i\}$, then we can write $D = 1 \otimes \mathrm{d} - \sum_i u^i \otimes \iota_{X_i}$, often shortened to $D = \mathrm{d} - \sum_i u^i \iota_{X_i}$. For $T = S^1$, this simplifies even further to fixing $X \in \mathfrak{s}^1$ and dual element u, with $D = \mathrm{d} - u\iota_X$.

3.3.2 Equivariant Cohomology at a Point

The equivariant cohomology for a 0-dimensional manifold $\{p\}$ is quite simple, because $\Omega^0(\{p\}) \cong \mathbb{R}$ and $\Omega^k(\{p\}) = \{\}$ for $k > 0$. Thus, $\Omega_G(\{p\}) = S(\mathfrak{g}^*)^G$, and as $D = 0$, every form is equivariantly closed and none are equivariantly exact, meaning $H^*_G(\{p\}) = S(\mathfrak{g}^*)^G$ as well.

3.3.3 Equivariant Cohomology of a Free Group Action

Now, we know that if G is a compact, connected Lie group acting freely on M then M/G is a smooth manifold as well. We would like to see if our definition of $H^\bullet_G(M)$ corresponds with $H^\bullet(M/G)$, as would seem to be prudent.

Theorem 3.3.6. *Let M have a free G-action, where G is a compact, connected Lie group.*

Then

$$H^\bullet_{\mathrm{d.R.}}(M/G) \cong H^\bullet(\Omega_G(M), D).$$

Proof. Note that $\pi : M \to M/G$ is a principal bundle (Def. (4.2.4)), so $\pi^* : \Omega(M/G) \to \Omega(M)$ is injective [Tu11, Prob. 18.8]. Additionally, note that elements of $\pi^*\Omega(M/G)$ are necessarily G-invariant. We will now construct an isomorphism between $H^\bullet(\Omega_G(M), D)$ and $H^\bullet(\Omega(M/G), \mathrm{d})$. First, consider the map $i : \Omega(M/G) \to \Omega_G(M)$ defined by $i(\omega) = 1 \otimes (\pi^*\omega)$. We have that

$$\begin{aligned}
(Di(\omega))(X) &= (D(1 \otimes \pi^*\omega))(X) \\
&= \mathrm{d}\pi^*\omega + \iota_X \pi^*\omega \\
&= \mathrm{d}\pi^*\omega + \pi^*(\iota_{\pi_* X^\#}\omega) \\
&= \pi^*\mathrm{d}\omega \\
Di(\omega) &= 1 \otimes \pi^*\mathrm{d}\omega = i(\mathrm{d}\omega),
\end{aligned}$$

where we used that $\pi_* X^\# = 0$, as G acts trivially on M/G.

Thus, as i is compatible with addition and scalar multiplication as well, it is a cochain homomorphism. Now, we need to show that it is also cohomological isomorphism, that is, that every cohomology class in $H^\bullet(\Omega_G(M), D)$ corresponds to one in $H^\bullet(M/G)$ and vice versa.

Let us consider the quotient complex $\Omega_G(M)/i(\Omega(M/G))$, i.e., the complex where

$$(\Omega_G(M)/i(\Omega(M/G)))^k := (\Omega^k_G(M)/i(\Omega^k(M/G))).$$

We have that $D(i(\Omega^k(M/G)) \subseteq i(\Omega^{k+1}(M/G))$, as $D(i(\pi^*\varpi)) = i(\pi^*(\mathrm{d}\varpi))$, so D respects this grading on $\Omega_G(M)/i(B)$, and as $D^2 = 0$, it forms a proper cochain complex. We then have the short exact sequence of cochain complexes

$$0 \longrightarrow (\Omega(M/G), \mathrm{d}) \xrightarrow{\ i\ } (\Omega_G(M), D) \xrightarrow{\ p\ } \left(\frac{\Omega_G(M)}{i(\Omega(M/G))}, D\right) \longrightarrow 0.$$

This then induces a long exact sequence on the cohomology by the zig-zag lemma [Tu11, Thm. 25.6]

$$\cdots \longrightarrow H^k(B, \mathrm{d}) \xrightarrow{\ i^*\ } H^k(\Omega_G(M), D) \xrightarrow{\ p^*\ } H^k\left(\frac{\Omega_G(M)}{i(\Omega(M/G)))}, D\right) \longrightarrow \cdots.$$

Thus, to show that i^* is an isomorphism, we need to show that the cohomology of $\Omega_G(M)/i(\Omega(M/G)))$ is 0, as then by definition,

$$H^k(\Omega_G(M), D) = \ker p^* = i^*(H^k(\Omega(M/G), \mathrm{d})),$$

which combined with the injectivity of i, produces an isomorphism on the cohomology.

To do this, first note that as $\pi : M \to M/G$ is a principal bundle, we can construct a connection Θ (Def. (4.2.5)). Further, we can construct a derivation $K : S(\mathfrak{g}^*) \otimes \Omega(M)$ such that for all $\alpha \in \mathfrak{g}^*$, $K(\alpha) = -\alpha \circ \Theta \in \Omega^1(M)$, and $K(1 \otimes \Omega(M)) = 0$.

Then, if $\beta = u^I \beta_I \in (S^k(\mathfrak{g}^*) \otimes \Omega(M))^G$, we have that

$$\begin{aligned} K(\beta) &= \sum_{i \in I} K(u^i) u^{I \setminus \{i\}} \beta_I \\ &= -\sum_{i \in I} u^{I \setminus \{i\}} (\Theta^i \wedge \beta_I) \in (S^{k-1}(\mathfrak{g}^*) \otimes \Omega(M))^G, \end{aligned}$$

where we are treating I as a multiset and subtracting $\{i\}$. The above holds, as

$$\mu_{g^{-1}}^* \Theta^i = \mu_{g^{-1}}^*(u^i \circ \Theta) = u^i \circ (\mu_{g^{-1}}^* \Theta)$$

$$= u^i \circ (\mathrm{Ad}_{g^{-1}} \Theta) = (u^i \circ \mathrm{Ad}_{g^{-1}}) \circ \Theta,$$

so

$$g \cdot K(\beta) = -\sum_{i \in I} \left(u^{I \setminus \{i\}} \circ \mathrm{Ad}_{g^{-1}}\right) (((u^i \circ \mathrm{Ad}_{g^{-1}}) \circ \Theta^i) \wedge \beta_I)$$

$$= K(g \cdot \beta) = K(\beta).$$

Furthermore, if $\omega \in \Omega_G^m(M)$, $K(\omega) \in \Omega_G^{m-1}(M)$, as u^i gets sent to a 1-form, decreasing degree by 1, and a pure differential form gets sent to 0, which also can be viewed as having one less degree, as 0 has every degree.

Now, note that if $\alpha \in \mathfrak{g}^*$,

$$((DK + KD)\alpha)(X) = (DK\alpha)(X)$$

$$= (-D(\alpha \circ \Theta))(X)$$

$$= -\mathrm{d}(\alpha \circ \Theta) + \iota_X(\alpha \circ \Theta)$$

$$= \alpha(X) - \mathrm{d}(\alpha \circ \Theta),$$

as $\iota_X(\alpha \circ \Theta) = \alpha \circ (\iota_X \Theta) = \alpha(X)$ by definition. Thus, $(DK + KD)\alpha = (\mathrm{polydeg}\alpha)\alpha + [0]$, where $\mathrm{polydeg}\alpha$ refers to α's polynomial degree, and $[n]$ is a term of polynomial degree n or lower.

Assume that γ_1, γ_2 have monomial polynomial components of degrees k_1 and k_2 respectively, and each individually satisfy $(DK + KD)\gamma_1 = k_1\gamma_1 + [k_1 - 1]$ and $(DK + KD)\gamma_2 =$

$k_2\gamma_2 + [k_2 - 1]$. Then we have that

$$(DK + KD)(\gamma_1 \wedge \gamma_2) = ((DK + KD)\gamma_1) \wedge \gamma_2 + \gamma_1 \wedge (DK + KD)\gamma_2$$

$$= (k_1\gamma_1 + [k_1 - 1]) \wedge \gamma_2 + \gamma_1 \wedge (k_2\gamma_2 + [k_2 - 1])$$

$$= (k_1 + k_2)\gamma_1 \wedge \gamma_2 + [k_1 + k_2 - 1],$$

as $(DK + KD)$ is a derivation. As $\text{polydeg}(\gamma_1 \wedge \gamma_2) = k_1 + k_2$, this degree-lowering property is held by $\gamma_1 \wedge \gamma_2$ as well. Thus, by induction, we have that the degree-lowering property holds for every equivariant differential form with monomial polynomial component.

This in turn implies that it holds for a non-monomial element of $\Omega_G^k(M)$, as if its polynomial degree was n, then all monomial elements γ of degree a less than n satisfy $(KD + DK)\gamma = a\gamma + [a - 1] = n\gamma + (n - a)\gamma + [a - 1] = n\gamma + [n - 1]$.

Suppose we have an element $\beta \in \Omega_G^k(M)$ with maximum polynomial degree n such that $D\beta \in i(\Omega(M/G))$. Then, as we showed above,

$$(DK + KD)\beta = n\beta + \gamma = n\beta + [n - 1].$$

Now, as $D\beta \in i(\Omega(M/G))$, this means that

$$DK\beta = n\beta + \gamma$$

$$\beta = DK\beta - \gamma/n.$$

Note, that K preserves $\Omega_G(M)$, and that K respects its graded structure, as does D, so $-\gamma/n \in \Omega_G^k(M)$, $D(-\gamma/n) = D\beta \in i(\Omega(M/G))$, and the maximum degree of $-\gamma/n$, n_1, is less than n. Thus, one can show that $DK(-\gamma/n) = (-\gamma/n) - \gamma_2/n_1$, with $-\gamma_2/n_1$ being a k-cocycle of polynomial degree $\leqslant n_1$, and so on. Thus, eventually we reach that $\beta = D(\text{stuff}) + 1 \otimes \varpi$, with $\varpi \in (\Omega^k(M))^G$ such that $D(i(\varpi)) \in i(\Omega(M/G))$. Note that if

$D(i(\varpi)) = i(\alpha)$ for some $\alpha \in \Omega(M/G)$, then

$$\mathrm{d}\varpi - \iota_X \varpi = \alpha.$$

Thus, $\iota_X \varpi$ is completely X-invariant, which is only possible if $\iota_X \varpi = 0$ for all $X \in \mathfrak{g}$. Thus, ϖ is both G-invariant and horizontal, meaning it is basic, i.e. that $\varpi \in \pi^* \Omega(M/G)$ [GS99, p. 26]. Thus, any element $\beta \in \Omega_G(M)$ is cohomologous to an element of $i(\Omega(M/G))$, so $\Omega_G(M)/i(B)$ has a trivial cohomology, proving that i gives a quasi-isomorphism.

□

Chapter 4: ABBV Localization

Now, we come to the climax of this thesis: the application of the theory of equivariant cohomology to calculating integrals. Once we develop an appropriate definition for integration of equivariant forms, we will go on to state a fascinating theorem that shows integrals of equivariantly closed forms can be entirely characterized by integrals of their restrictions to the fixed-point submanifold, a result evocative of the residue theorem of complex analysis.

4.1 Integrating Equivariant Forms

Naturally, one may begin to wonder if there is a way to integrate equivariant differential forms analagous to how one may integrate a normal differential form. The answer is a resounding "yes."

Definition 4.1.1. Given $\alpha \in \Omega_G(M)$, we define the polynomial function $\int_M \alpha : \mathfrak{g} \to \mathbb{R}$ by

$$\left(\int_M \alpha\right)(X) \equiv \int_M \alpha(X).$$

In other words, given $\alpha = \sum_I u^I \alpha_I$,

$$\int_M \alpha = \sum_I u^I \int_M \alpha_I.$$

Notice that for this integral, as $\int_M \alpha_I = 0$ unless $\alpha_I \in \Omega^n(M)$, we only need to care about terms which have degree-n differential forms multiplied to polynomials. Furthermore, notice that

$$\begin{aligned}\left(\int_M \alpha\right)(\mathrm{Ad}_g(X)) &= \int_M \alpha(\mathrm{Ad}_g(X)) \\ &= \int_M \mu^*_{g^{-1}}(\alpha(X)) \\ &= \int_M \alpha(X),\end{aligned}$$

where we used Thm. (2.3.12) to get to the last line. Thus, $\int_M \alpha \in S(\mathfrak{g}^*)^G$.

4.1.1 G-actions on submanifolds and boundaries

Here we note a few things about how group actions interact with submanifolds.

Definition 4.1.2 (G-invariant submanifold)**.** A smooth submanifold S is G-invariant if for all $p \in S$, $\mu_g p \in S$.

Now, consider a smooth G-manifold M with boundary ∂M. Then, as μ_g is a diffeomorphism for all $g \in G$, $\mu_g(\partial M) = \partial M$ [Lee12, Prop. 2.18], so ∂M is a G-invariant submanifold, meaning that the group action restricts to it.

4.1.2 Equivariant Stokes' Theorem

We have an almost identical statement as the standard Stokes' theorem for equivariant forms.

Theorem 4.1.3. *Let M be an oriented smooth n-dimensional G-manifold. Then, for $\alpha \in \Omega_G(M)$ such that $\alpha(X)$ has compact support for all $X \in \mathfrak{g}$,*

$$\int_M D\alpha = \int_{\partial M} \alpha \tag{4.1}$$

Proof. Using the definition of equivariant integration, we realize that we can just consider $\int_M D\alpha(X)$ and $\int_{\partial M} \alpha(X)$ for arbitrary $X \in \mathfrak{g}$. This in turn means that $\int_M D\alpha$ will only be nonzero if it has a term in $(S(\mathfrak{g}^*) \otimes \Omega^{n-1}(M))^G$, and without loss of generality, we need only consider elements of that subset. Using the definition of the Cartan differential, we have that

$$\begin{aligned}\int_M (D\alpha(X)) &= \int_M (\mathrm{d}(\alpha(X)) - \iota_X(\alpha(X))) \\ &= \int_M \mathrm{d}(\alpha(X)),\end{aligned}$$

as $\alpha(X) \in \Omega^{n-1}(M)$, so $\iota_X \alpha(X) \in \Omega^{n-2}(M)$ and thus integrates to 0 on M. Then, we can use the standard Stokes' theorem to say that

$$\begin{aligned}\left(\int_M D\alpha\right)(X) &= \int_M D\alpha(X) \\ &= \int_M \mathrm{d}(\alpha(X)) \\ &= \int_{\partial M} \alpha(X) \\ &= \left(\int_{\partial M} \alpha\right)(X),\end{aligned}$$

which, as it holds for all $X \in \mathfrak{g}$, proves the assertion above. $\square$

4.2 ABBV Localization

Now, we finally get to stating some localization theorems. But first, a few more concepts must be introduced.

4.2.1 The Normal Bundle

Normal bundles are a key ingredient of the statement of our localization theorems. Intuitively (and in the case of Riemannian manifolds, exactly), they capture the directions orthogonal to a submanifold in the tangent space.

Definition 4.2.1 (Restriction of the tangent bundle)**.** Let $S \subseteq M$ be a subset of a smooth n-manifold M. Then we define $TM|_S$ to be $\sqcup_{p \in S} T_pM$. If S is a smooth submanifold of M, then $TM|_S$ is a smooth vector bundle [Lee12, Ex. 10.8].

Let N be a submanifold to M. Then there is a natural identification of TN as a subbundle of $TM|_N$. The normal bundle is the pointwise complement to that subbundle.

Definition 4.2.2 (Normal bundle)**.** Let $S \subseteq M$ be a smooth submanifold of n-manifold M. Then the normal bundle to S NS is the bundle given by $N_pS \equiv T_pM/T_pS$ for all $p \in S$.

NS is a smooth vector bundle.

4.2.2 Equivariantly Closed Extensions

Recall that an equivariant form α is equivariantly closed if $D\alpha = 0$. Given some invariant $2n$-form ω_{2n}, we say that it has an equivariantly closed extension if one can construct an equivariantly closed form α such that $\alpha(0) = \omega_{2n}$. In other words, for all $X \in \mathfrak{g}$,

$$(D\alpha)(X) = \mathrm{d}\alpha(X) - \iota_X \alpha(X) = 0$$

$$\mathrm{d}\alpha(X) = \iota_X \alpha(X).$$

Note that any equivariantly closed extension of ω_{2n} has degree $2n$ as an equivariant form. Let $[\alpha(X)]_k$ denote the degree-k component of $\alpha(X)$. Then,

$$\mathrm{d}[\alpha(X)]_{2n} + \mathrm{d}[\alpha(X)]_{2n-2} + \cdots + \mathrm{d}[\alpha(X)]_0 = \iota_X[\alpha(X)]_{2n} + \cdots + \iota_X[\alpha(X)]_2.$$

For a form to be 0 each degree must be 0, so we have that

$$\mathrm{d}[\alpha(X)]_{2(n-k)} = \iota_X[\alpha(X)]_{2(n-k+2)}$$

for $k = 0, 1, \ldots, k$. This clearly means that ω_{2n} must be closed in order for it to admit an equivariantly closed extension.

Unfortunately, while there has been some investigation into which closed G-invariant forms admit an equivariantly closed extension [Wu93], there does not exist, or at least has not yet been found, a set of comprehensive necessary and/or sufficient conditions that is, in my opinion, truly elegant.

G-manifolds for which every closed form admits an equivariantly closed extension are called equivariantly formal.

4.2.3 The Equivariant Euler Class

The equivariant Euler class is the final ingredient we have to unpack in order to make sense of our localization theorems. It is an equivariant generalization of the Euler class, a characteristic class which intuitively captures the nontriviality of a vector bundle.

Definition 4.2.3 (Equivariant vector bundle)**.** A vector bundle $\pi : E \to M$ with fiber V is G-equivariant if E, and M have G-actions μ and ν such that $\mu_g(E_p) \subseteq E_{\nu_g p}$ and $\mu_g|_{E_p} : E_p \to E_{\nu_g p}$ is a linear map. In other words, if $\pi : E \to M$ is an equivariant vector bundle, π is G-equivariant, and μ restricts to linear maps on individual fibers.

Note that if S is a G-invariant submanifold, then NS is an equivariant vector bundle, as one can restrict μ_g to S and consider the pushforward, thereby getting a map for $p \in S$ $(\mu_g)_* : T_pM/T_pS \to T_{\mu_g p}M/T_{\mu_g p}S$.

We now introduce a number of concepts from the theory of principal bundles we will make use of in our definition of the equivariant Euler class.

Definition 4.2.4 (Principal bundle)**.** Given a Lie group G, a principal G-bundle $\pi : P \to M$ is a smooth fiber bundle equipped with a smooth free right G-action R_g that restricts to a transitive action on the fibers, i.e., for all $p \in M$ and $g \in G$, R_g sends elements of $\pi^{-1}(\{p\})$ to $\pi^{-1}(\{p\})$, and $R|_{\pi^{-1}(\{p\})} : g \to R_g|_{\pi^{-1}(\{p\})}$ is transitive for all $p \in M$.

Note that this definition implies that $\pi^{-1}(\{p\})$ is diffeomorphic to G for all $p \in M$, i.e., P has fiber G. Thus, we can think of $\pi^{-1}(\{p\})$ as G, just without a chosen identity.

Definition 4.2.5 (Connection on a principal bundle)**.** Let $\pi : P \to M$ be a principal G-bundle. Then, a connection on P is a $\mathfrak{g}$-valued one-form $\Theta \in \mathfrak{g} \otimes \Omega^1(P)$ satisfying the following properties: $R_g^*\Theta = \mathrm{Ad}_{g^{-1}}\Theta$, and for all $X \in \mathfrak{g}$, $\iota_{X^\#}\Theta = X$.

Note that for a right-action, the fundamental vector field is defined in much the same way as for a left-action, with $(X^\#)_p = R(p)_*X$, where $R(p) : G \to M$ is defined by $R(p)(g) = R_g(p)$.

Definition 4.2.6 (Curvature of a connection)**.** Given a principal G-bundle $\pi : P \to M$ and a connection Θ, the curvature of the connection, $\Omega \in \mathfrak{g} \otimes \Omega^2(P)$ is defined as

$$\Omega := \mathrm{d}\Theta + \frac{1}{2}[\Theta, \Theta],$$

where given $\alpha, \beta \in \mathfrak{g} \otimes \Omega^1(P)$, $[\alpha, \beta]$ is the $\mathfrak{g}$-valued 2-form defined by

$$[\alpha, \beta](X, Y) = [\alpha(X), \beta(Y)] - [\alpha(Y), \beta(X)]$$

for any vector fields X, Y.

Note that this definition just simplifies to stating that

$$\Omega(X, Y) = \mathrm{d}\Theta(X, Y) + [\Theta(X), \Theta(Y)].$$

A connection is called flat if has 0 curvature.

Theorem 4.2.7. *A trivial principal G-bundle admits a flat connection.*

Proof. Recall a trivial bundle P is isomorphic to $M \times G$, with $\pi : M \times G \to M$. Thus, $TP \cong TM \oplus TM$. Let $\theta \in \mathfrak{g} \otimes \Omega^1(G)$ be the Maurer-Cartan form, that is, the uniqe $\mathfrak{g}$-valued one form such that $\theta_g((\ell_g)_*X) = X$. Let $\pi_2 : M \times G \to G$ be the projection onto G. Then, consider $\Theta = \pi_2^*\theta$. This is a connection on P. Now, consider the curvature Ω.

Clearly, if $X \in TM$ $\Omega(X,Y) = 0$ for all $Y \in TP$. But, we also know that if $X, Y \in TG$ then $\Omega(X,Y) = 0$ [Tu20, Thm. 17.4]. Thus, we have that Ω is identically 0, so Θ is a flat connection on P. □

Theorem 4.2.8. *If G is a compact, connected Lie group acting smoothly and equivariantly on a vector bundle $\pi : E \to M$, then we can endow E with a G-invariant bundle metric, i.e. a bundle metric such that for all $X, Y \in E_p$, $\langle X, Y\rangle_p = \langle g \cdot X, g \cdot Y\rangle_{g\cdot p}$.*

The specific principal bundle we will use is the following.

Definition 4.2.9 (The frame bundle)**.** Given a real oriented rank-r vector bundle E over M with bundle metric $\langle\cdot,\cdot\rangle$, the frame bundle is defined as

$$\mathcal{F}(E) := \bigsqcup_{p\in M} \mathcal{F}(E_p),$$

where $\mathcal{F}(E_p)$ is the space of oriented orthonormal bases of T_pM.

$\mathcal{F}(E)$ is a principal $SO(r)$-bundle. Note that if E is a G-equivariant vector bundle with left G-action, then $\mathcal{F}(E)$ also has a left G-action which commutes with its right $SO(r)$-action. Thus we can endow $\mathcal{F}(E)$ with a G-invariant connection by taking any arbitrary connection θ on $\mathcal{F}(E)$ and letting $\Theta = \int_G (\mu_g^*\theta)\mathrm{d}g$, where we are integrating over the normalized Haar measure on G. It is straightforward to show this is G-invariant and a connection.

Definition 4.2.10 (The Pfaffian)**.** Given a $2n$-dimensional inner product space V, the Pfaffian is a $SO(V)$-invariant polynomial map defined on $\mathfrak{so}(V)$ so that for $A = (A_{ij}) \in \mathfrak{so}(V)$, if e^i is a dual basis for V, and

$$\omega := \sum_{i<j} A_{ij} e^i \wedge e^j,$$

$$\frac{\omega^{\wedge n}}{n!} = \text{Pfaff}(A) e^1 \wedge e^2 \wedge \cdots \wedge e^{2n}$$

Explicitly, for a 2×2 skew-symmetric matrix $A = \begin{pmatrix} 0 & a \\ -a & 0 \end{pmatrix} \in \mathfrak{so}(2)$, $\text{Pfaff}(A) = a$. Additonally, it is interesting to note that $\text{Pfaff}(A)^2 = \det A$.

Finally, we are able to define the equivariant Euler class.

Definition 4.2.11 (Equivariant Euler class)**.** Given a G-equivariant real rank-$2k$ oriented vector bundle E, and a G-invariant connection Θ on $\mathcal{F}(E)$ with respective curvature Ω, the equivariant Euler class of E is an element of $H_G^{2k}(M)$ defined by

$$e^G(E) := \left[\frac{1}{(2\pi)^k} \text{Pfaff}\left(\Omega + L\right)\right],$$

where $L : \mathfrak{g} \to \Omega^0(M)$ is the map defined by $L(X) = \iota_X \Theta$.

Note that this definition has an opposite sign on the second argument inside the Pfaffian than [BT01], as that paper uses opposite sign conventions for the fundamental vector field of left-acting G-manifolds than we do. We will now state some facts about the equivariant Euler class. Actually proving them is outside the scope of this thesis, but one can do it if one uses an alternative definition of the equivariant Euler class based on the Borel model of equivariant cohomology, wich [BT01] showed to be equivalent.

Theorem 4.2.12. *The equivariant Euler class does not depend on choice of connection Θ, nor on G-invariant bundle metric.*

Theorem 4.2.13. *The equivariant Euler class has the following properties*

(a) If $\pi_1 : E \to M$ and $\pi_2 : F \to M$ are real, oriented, G-equivariant vector bundles, then $e^G(E \oplus F) = e^G(E) \wedge E^G(F)$.

(b) If E and $\bar{E}$ are the same vector bundle with opposite orientations, then $e^G(E) = -e^G(\bar{E})$.

It is worth noting that $e^G(E)$ also gives an equivariantly closed extension of the standard Euler class.

We now calculate the equivariant Euler class for a specific case we will make use of later.

Theorem 4.2.14. *Suppose $\pi : L^m \to \{p\}$ is a trivial rank-2 real S^1-equivariant oriented vector bundle, with $\mu_{e^{it}} x = \begin{pmatrix} \cos mt & -\sin mt \\ \sin mt & \cos mt \end{pmatrix} x$ in some properly-oriented basis. Then $e^{S^1}(L^m) = -\frac{mu}{2\pi}$, where u is dual to i, in that if X satisfies $\exp(X) = e^i$, $u(X) = 1$.*

Proof. Notice that $\pi : L^m \to \{p\}$ is a trivial vector bundle. Furthermore, notice that the standard inner product on $\mathbb{R}^2$ is S^1-invariant if we are in the basis specified in the theorem statement. As the equivariant Euler class does not depend on choice of S^1-invariant inner product, let us choose that inner product. Thus, we can construct an orthonormal frame bundle $\mathcal{F}(L^m)$. This is a S^1-equivariant $SO(L^m) \cong SO(2)$ principal bundle, with a left S^1-action corresponding to

$$(v_1, v_2) \to (\mu_g \mathbf{v}_1, \mu_g \mathbf{v}_2)$$

and a right $SO(2)$ action corresponding to the matrix multiplication of $(\mathbf{v}_1 \mathbf{v}_2) \to (\mathbf{v}_1\ \mathbf{v}_2)A$. Note that $(\mathbf{v}_1\ \mathbf{v}_2) \in SO(2)$, and the matrix corresponding to the linear map μ_g is in $SO(2)$ as well by the S^1-invariance of the inner product. As such, by the commutativity of $SO(2)$, we can view $\mu_{e^{it}}$ as acting by right-multiplication by elements of $SO(2)$ as well.

Now let Θ be a flat S^1-invariant connection on $\mathcal{F}(L^m)$ (note that one must exist, as $SO(2)$ is abelian, and thus every connection is $SO(2)$ invariant, and as described above, we can view the S^1 action as taking place within the $SO(2)$ action). Then, we have that for $X \in \mathfrak{s}^1$ and corresponding dual element $u \in (\mathfrak{s}^1)^*$ as described in the theorem statement, $e^{S^1}(L^m) = \frac{1}{2\pi}\mathrm{Pfaff}(\Omega + uL_X)$. As Θ is flat, $\Omega = 0$, so we need only consider L_X. Notice that as $L_X = \iota_X \Theta$, $L_X \in \mathfrak{so}(2)$. Now, we need to characterise $X^\#$ for $X \in \mathfrak{s}^1$. Note that $X^\#$ acts on $f \in C^\infty(\mathcal{F}(L^m))$ as

$$\begin{aligned}
(X^\# f)((\mathbf{v}_1\ \mathbf{v}_2)) &= \frac{\mathrm{d}}{\mathrm{d}t}\Big|_0 f(\mu_{e^{it}}(\mathbf{v}_1\ \mathbf{v}_2)) \\
&= \frac{\mathrm{d}}{\mathrm{d}t}\Big|_0 f\left(\begin{pmatrix} \cos mt & -\sin mt \\ \sin mt & \cos mt \end{pmatrix}(\mathbf{v}_1\ \mathbf{v}_2)\right) \\
&= \frac{\mathrm{d}}{\mathrm{d}t}\Big|_0 f\left((\mathbf{v}_1\ \mathbf{v}_2)\begin{pmatrix} \cos mt & -\sin mt \\ \sin mt & \cos mt \end{pmatrix}\right) \\
&= \frac{\mathrm{d}}{\mathrm{d}t}\Big|_0 f\left((\mathbf{v}_1\ \mathbf{v}_2)\exp\begin{pmatrix} 0 & -mt \\ mt & 0 \end{pmatrix}\right) \\
&= \left(\begin{pmatrix} 0 & -m \\ m & 0 \end{pmatrix}^\# f\right)((\mathbf{v}_1\ \mathbf{v}_2)).
\end{aligned}$$

Thus, we see that $L_X = \iota_X \Theta = \begin{pmatrix} 0 & -m \\ m & 0 \end{pmatrix}$, as for $Y \in \mathfrak{so}(2)$, $\Theta(Y^\#) = Y$ by the definition of a connection. Thus, we have

$$e^{S^1}(L^m) = \frac{1}{2\pi}\mathrm{Pfaff}\left(u\begin{pmatrix} 0 & -m \\ m & 0 \end{pmatrix}\right) = -\frac{um}{2\pi}$$

by definition of the Pfaffian of a 2×2 skew-symmetric matrix. □

4.2.4 ABBV Localization

Finally, we are able to state the localization theorem in concrete terms.

Theorem 4.2.15 (Atiyah-Bott [AB84] and Berline–Vergne [BV82])**.** *Let T be a torus, and*

M be a compact smooth T-manifold, with fixed-point manifold $M^T = \cup_i F_i$ with connected components $\{F_i\}_I$. For each F_i let $\iota_{F_i} : F_i \hookrightarrow M$ be its inclusion map. Let ϕ be an equivariantly closed form. Then,

$$\int_M \phi = \sum_{i \in I} \int_{F_i} \frac{\iota_{F_i}^* \phi}{e^T(NF_i)},$$

where NF_i is the normal bundle of f, and $e^T(\cdot)$ is the T-equivariant Euler class.

Note that for all $t \in T$, $\mu_t|_F = \mathrm{id}_f$, so for all F_i, $\Omega_T(F_i) = \Omega(f) \otimes S(\mathfrak{t}^*)$. Also note that if M^T consists of isolated points, $NF_i = \{T_{F_i}M \to F_i\}$ is a trivial bundle, $\Omega_T(F_i) = S(\mathfrak{t}^*)$, and the integral disappears, leaving just

$$\int_M \phi = \sum_{i \in I} \frac{\iota_{F_i}^* \phi}{e^T(TM|_{F_i})}.$$

One may ask how exactly you divide by an equivalence class of equivariant differential forms, as is implied by dividing by the equivariant Euler class. The answer is that as $H_T^*(F_i) = S(\mathfrak{t}^*) \otimes H^*(F_i)$, it is a $S(\mathfrak{t}^*)$-module, so we can perform an algebraic operation called localization.

Localization is the algebraic process by which, given some $f \in S(\mathfrak{g}^*)$, one constructs a $\mathbb{R}[u_1, u_2, \ldots, u_n][f^{-1}]$-module N_f out of a $\mathbb{R}[u_1, u_2, \ldots, u_n]$-module N (we are switching to lower-indexing of us for convenience). In this case, we choose f in the following way: each equivariant Euler class can be written as

$$e^T(NF_i) = \alpha_{2k;i} + \left(\sum_{\{j_1\}} \alpha_{2k-2;i}^{\{j_1\}} u_{j_1} \right) + \cdots + \left(\sum_{\{j_1, \ldots, j_k\}} \alpha_{0;i}^{\{j_1, \ldots, j_k\}} u_{j_1} \cdots u_{j_k} \right).$$

The last sum, as an element of $H^0(M) \otimes S(\mathfrak{g}^*)^G \cong \mathbb{R} \otimes S(\mathfrak{g}^*)^G$, is just a polynomial. Let P_i be the last, purely polynomial, term of $e^T(NF_i)$. If $P_i \neq 0$ for all i, which is the case for the equivariant Euler class of a connected component of M^T, which F_i is by definition we can localise over $\prod_i P_i$. Note that P_i is invertible in this localization, with $P_i^{-1} = \frac{\prod_{j \neq i} P_j}{\prod_j P_j}$ [GS99, Thm. 10.8.1]. Furthermore, we can invert $e^T(NF_i)$, letting $e^T(NF_i)$ have a ring-structure induced by the wedge product. To do this, we note that as

$$e^T(NF_i) = P_i \left(1 + \frac{\alpha_{2k;i}}{P_i} + \cdots + \frac{\left(\sum_{\{j_1, j_2, \ldots, j_{k-1}\}} \alpha_{2;i}^{\{j_1, j_2, \ldots, j_{k-1}\}} u_{j_1} \cdots u_{j_{k-1}} \right)}{P_i} \right),$$

if we let

$$\beta_i \equiv - \left(\frac{\alpha_{2k;i}}{P_i} + \cdots + \frac{\left(\sum_{\{j_1, j_2, \ldots, j_{k-1}\}} \alpha_{2;i}^{\{j_1, j_2, \ldots, j_{k-1}\}} u_{j_1} \cdots u_{j_{k-1}} \right)}{P_i} \right),$$

then $e^T(NF_i) = P_i(1 - \beta_i)$. Now, we can invert this with the aid of power series, noting that

$$\frac{1}{e^T(NF_i)} = \frac{1}{P_i} \frac{1}{1 - \beta_i} = \frac{1}{P_i} \left(1 + \beta_i + \beta_i \wedge \beta_i + \cdots + \beta_i^{\wedge n} + \cdots \right).$$

However, as $\Omega^{2n}(F_i) = 0$ for $n \geqslant \dim M \geqslant \dim F_i$, this series will terminate at $(\dim F_i)^{\text{th}}$, and thus also n^{th} order. It is simple to verify then that

$$e^T(NF_i) \wedge \frac{1}{e^T(NF_i)} = 1,$$

where 1 is the cohomology class of the constant map $p \to 1$, the multiplicative identity in the cohomology ring.

An element of $H_T^*(F_i)$ like $\iota_{F_i}\phi$ has a natural inclusion into $H_T^*(F_i)_{\prod_i P_i}$ just by dividing it through by $(\prod_i P_i)^0 = 1$. Thus, one of the assertions of the localization theorem above is that the fractional parts of each of these terms cancel out in some way, leaving us with just an element of $S(\mathfrak{t}^*)$.

The Localization Formula for an S^1-Action with Isolated Fixed Points

In the case of $T = S^1$, if there are isolated fixed points, we can do quite a bit more. Notice that for any $p \in M^{S^1}$, $Np = T_pM$.

Furthermore, we have the following fact.

Theorem 4.2.16. *Suppose p is an isolated fixed point under an S^1-action. Then, under $(\mu_{e^{i\theta}})_*$, $T_pM = L^{m_1} \oplus L^{m_2} \oplus \cdots \oplus L^{m_{n/2}}$, where L^{m_k} is a 2-dimensional subspace acted upon by $(\mu_g)_*$ like so:* $\begin{pmatrix} \cos m_k\theta & -\sin m_k\theta \\ \sin m_k\theta & \cos m_k\theta \end{pmatrix}$.

We call the ms the exponents of the fixed point.

Corollary 4.2.16.1. *If M is a connected S^1-manifold with isolated fixed points, then M is even-dimensional.*

Thus let us let M be $2k$-dimensional from now on. Then, the equivariant Euler class at an isolated fixed point $\{p\}$ becomes $e^{S^1}(L^{m_1}) \wedge \cdots \wedge e^{S^1}(L^{m_k})$, which is equal to

$$\left(-\frac{u}{2\pi}\right)^k \prod_{j=1}^{k} m_j.$$

Thus, our localization theorem becomes

Theorem 4.2.17. *Let M be a $2n$-dimensional compact S^1-manifold, with isolated fixed points $p \in F$. Then, for an equivariantly closed S^1-form ϕ, if $\iota_p : \{p\} \hookrightarrow M$ is the inclusion* *map for* p,

$$\int_M \phi = \left(-\frac{2\pi}{u}\right)^n \sum_{p\in F} \frac{\iota_p^*\phi}{\prod_{k=1}^n m_k(p)},$$

where $\{m_k(p)\}$ *are the exponents of* p*, and* u *is as defined in Thm. (4.2.14).*

The Duistermaat-Heckman Theorem

One of the chief applications of this theorem is in the case where M is a $2n$-dimensional symplectic manifold, i.e., that M is equipped with a 2-form ω which is non-degenerate, i.e. for all $p \in M$, given any $X \in T_pM$, if $\omega(X, Y) = 0$ for all $Y \in T_pM$, then $X = 0$. $e^T(NF_i)$ is invertible in the localization of the equivariant cohomology in the way described above, so we can localize integrals.

We can consider $\mu : T \to \text{Aut}T_pM \cong GL(T_pM)$ as a representation of T at the fixed points, and as every finite-dimensional representation of a compact group is completely reducible, we know it must decompose into a direct sum of irreducible representations. Furthermore, by Schur's Lemma, we know the only irreducible representations have complex-dimension 1. Consider a one-complex-dimensional representation (ρ, V) of T. Consider $H, G \in \mathfrak{t}$. $e^He^G = e^{H+G}$ by commutativity of T and thus $\rho(e^{H+G}) = \rho(e^H)\rho(e^G)$. Note that $|\rho(e^G)| = 1$ as we can endow V with an inner product such that $\langle \rho(e^G)v, \rho(e^G)w\rangle = \rho(e^G)^*\rho(e^G)\langle v, w\rangle = \langle v, w\rangle$ for all $v, w \in V$. In other words, $\rho(e^G)^*\rho(e^G) = |\rho(e^G)|^2 = 1$, so $|\rho(e^G)| = 1$. Thus, $\rho(e^G) = e^{if(\exp(G))}$, where $f : T \to \mathbb{R}$ satisfies $f(\exp(H + G)) = f(\exp(H)) + f(\exp(G))$. This defines a map $\alpha : \mathfrak{t} \to \mathbb{R}$ by precomposition with the exponential map. α is clearly linear, so $\alpha \in \mathfrak{t}^*$. As $t \to (\mu_t)_*$ decomposes into n such irreducible representations, at each fixed point p we have n such weights $\{\alpha_{i,p}\}_{i=1}^n$.

Corollary 4.2.17.1 (Duistermaat-Heckman [DH82])**.** *Let* (M, ω) *be a compact symplectic* $2n$*-dimensional* T*-manifold with equivariant symplectic* 2*-form* $\tilde{\omega} = \omega + \phi$*. Furthermore, suppose* M^T *consists of isolated fixed points. Then,*

$$\left(\int_M e^{\tilde{\omega}}\right)(X) = \int_M e^{\phi(X)}\frac{\omega^n}{n!} = (2\pi)^n \sum_{p\in M^T} \frac{e^{(\phi(p))(X)}}{\prod \alpha_{i,p}(X)},$$

where $\alpha_{i,p} \in \mathfrak{t}^*$ *are the weights of* μ_*.

There are a few things to note: While $e^{\tilde{\omega}}$ is not an equivariant differential form, each term in the power series expansion is, so we can still prove it with localization. Second, $\phi \in \Omega^0(M)^T \otimes S^1(\mathfrak{t}^*) \cong C^\infty(M) \otimes \mathfrak{t}^*$ is just a T-invariant map smooth map $M \to \mathfrak{t}^*$. We call it the moment map. Finally, in the case that $T = S^1$, this can be rephrased in the following way: if if is the moment map of ω, with $f \in C^\infty(M)$, (using that $\mathfrak{s}^1 \cong i\mathbb{R}$), then we have that

$$\int_M e^{itf}\frac{\omega^n}{n!} = \left(\frac{2\pi}{t}\right)^n \sum_{p\in M^{S^1}} e^{i\pi \mathrm{sgn} H_f(p)/4} e^{itf(p)} \sqrt{\frac{\det \omega_p}{\det H_f(p)}},$$

where $H_f(p)$ is the hessian, and $\mathrm{sgn} H_f(p)$ is the difference in the number of positive and negative eigenvalues of $H_f(p)$, called its signature. In other words, the first-order stationary phase approximation is exact.

It is also worth noting, as a historical aside, that the Duistermaat-Heckman formula was not first introduced in the context of equivariant cohomology, but was rather an impetus for Atiyah and Bott to produce their localization formula in the first place (I am unsure as to what role it played in Berline and Vergne's, as I cannot read French). It is thus a testament to their success that the Duistermaat-Heckman formula follows so readily from theirs.

Bibliography

[AB84] M. F. Atiyah and R. Bott. The Moment Map and Equivariant Eohomology. *Topology*, 23:1–28, 1984.

[Aud04] Michèle Audin. *Torus Actions on Symplectic Manifolds*. Birkhäuser, Basel, 2004.

[BGV92] Nicole Berline, Ezra Getzler, and Michèle Vergne. *Heat Kernels and Dirac Operators*. Springer-Verlag, 1992.

[BT01] Raoul Bott and Loring W. Tu. Equivariant Characteristic Classes in the Cartan model. *arXiv:math/0102001*, February 2001.

[BV82] Nicole Berline and Michele Vergne. Classes caracteristiques equivariantes. formule de localisation en cohomologie equivariante. *Comptes rendus de l'Académie des Sciences*, 295:539–541, 1982.

[DH82] J.J. Duistermaat and G.J. Heckman. On the variation in the cohomology of the symplectic form of the reduced phase space. *Inventiones mathematicae*, 69:259–268, 1982.

[GS99] Victor. Guillemin and Shlomo Sternberg. *Supersymmetry and Equivariant de Rham Theory*. Mathematics Past and Present. Springer Berlin, 1999.

[Hal15] Brian C. Hall. *Lie Groups, Lie Algebras, and Representations*. Springer International Publishing, 2015.

[Hat00] Allen Hatcher. *Algebraic Topology*. Cambridge University Press, Cambridge, 2000.

[Lee11] John M. Lee. *Introduction to Topological Manifolds*, volume 202 of *Graduate Texts in Mathematics*. Springer New York, New York, NY, 2011.

[Lee12] John M. Lee. *Introduction to Smooth Manifolds*. Springer New York, 2012.

[Ros21] Paolo Rossi. Equivariant and supersymmetric localization in QFT. January 2021. arXiv: 2101.09205.

[Sza00] Richard J. Szabo. *Equivariant Cohomology and Localization of Path Integrals*, volume 63 of *Lecture Notes in Physics Monographs*. Springer Berlin Heidelberg, 2000.

[Tu11] Loring W. Tu. *Introduction to Manifolds*. Springer New York, 2011.

[Tu17] Loring W. Tu. *Differential Geometry: Connections, Curvature, and Characteristic Classes*. Springer Cham, 2017.

[Tu20] Loring W. Tu. *Introductory Lectures on Equivariant Cohomology: (AMS-204)*. Princeton University Press, 2020.

[Wu93] Si-ye Wu. Cohomological Obstructions to the Equivariant Extension of Closed Invariant Forms. 1 1993.